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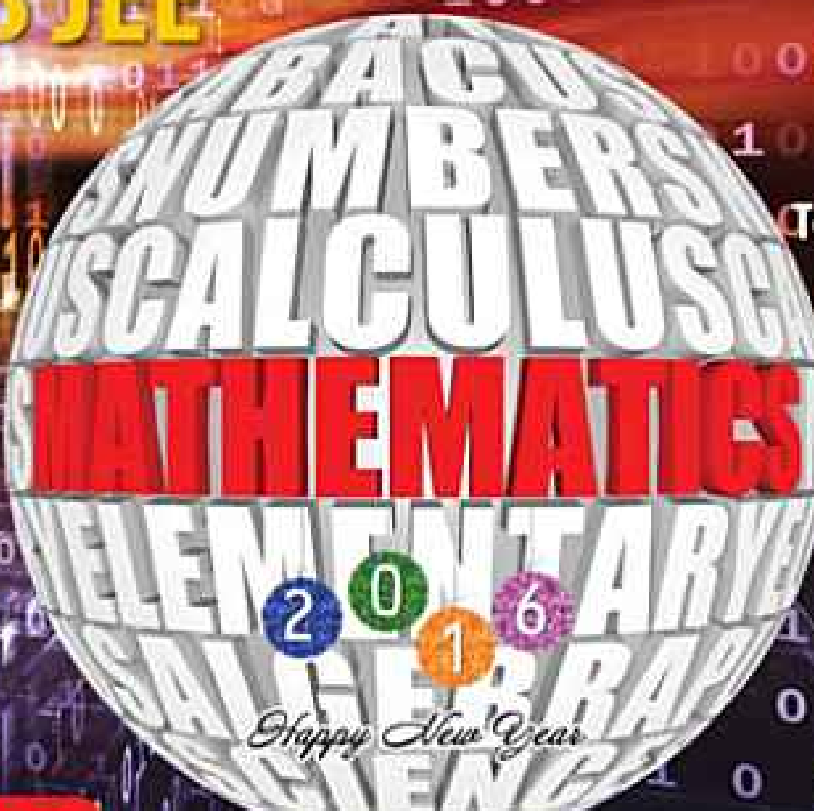

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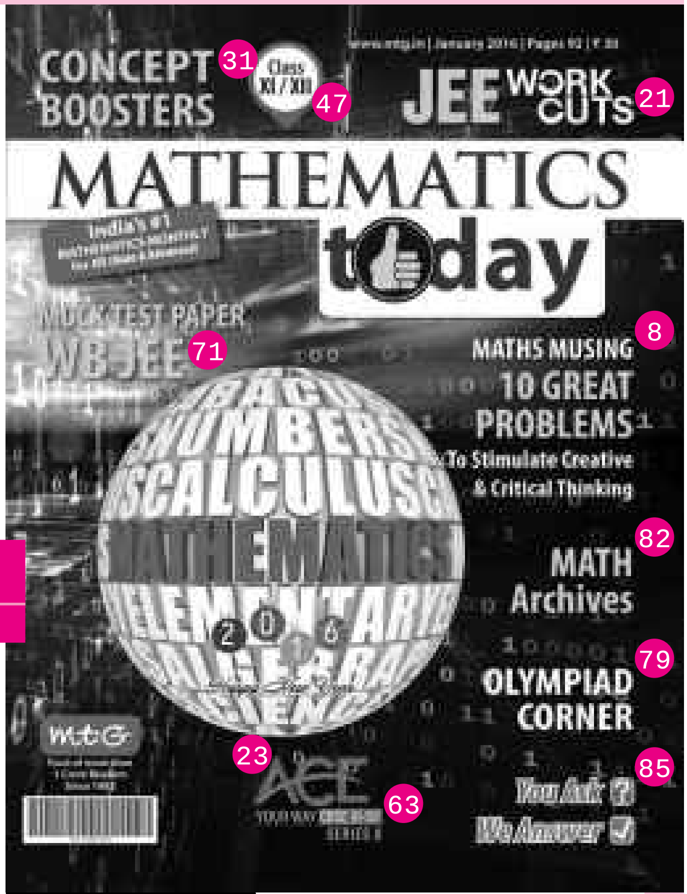

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Editor : Anil Ahlawat

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 157

JEE MAIN

1. Let f be twice differentiable such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = \{f(x)\}^2 + \{g(x)\}^2$, where $h(5) = 11$, then $h(10) =$
   (a) 11 (b) 10 (c) 9 (d) 8

2. The real values of a, the point $(-2a, a+1)$ will be an interior point of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$ is
   (a) $[-1, 0]$ (b) $(-1, -5+2\sqrt{6})$
   (c) $[0, 1]$ (d) $[-1, -5-2\sqrt{6}]$

3. A plane whose equation is $2x - y + 3z + 5 = 0$ is rotated through 90° about its line of intersection with the plane $5x - 4y - 2z + 1 = 0$. Find the equation of the plane in the new position.
   (a) $27x - 24y - 26z = 13$ (b) $27x + 24y + 26z = 13$
   (c) $27x - 24y + 26z = 13$ (d) none of these

4. The coefficient of x^{50} in the series
   $$S = \sum_{r=1}^{101} rx^{r-1}(1+x)^{101-r} \text{ is}$$
   (a) $\binom{100}{50}$ (b) $\binom{101}{50}$ (c) $\binom{102}{50}$ (d) $\binom{103}{50}$

5. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C, D respectively.
   If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, then the equation of the line is
   (a) $2x + 3y + 11 = 0$ (b) $2x + 3y + 22 = 0$
   (c) $x + 3y + 11 = 0$ (d) $2x + y + 22 = 0$

JEE ADVANCED

6. Two integers x and y are chosen (without random) at random from the set $\{x : 0 \le x \le 10, x \text{ is an integer}\}$ then the probability for $|x - y| \le 5$ is
   (a) $\frac{80}{121}$ (b) $\frac{90}{121}$ (c) $\frac{91}{121}$ (d) $\frac{85}{121}$

COMPREHENSION

Let $n \in N$. The A.M., G.M., H.M. and R.M.S. (root-mean square) of the n numbers $n+1, n+2, n+3, ..., n+n$ are A_n, G_n, H_n, R_n respectively. Then

7. $\lim_{n\to\infty} \frac{G_n}{n} =$
   (a) $\frac{1}{e}$ (b) $\frac{2}{e}$ (c) $\frac{3}{e}$ (d) $\frac{4}{e}$

8. $\lim_{n\to\infty} \frac{R_n}{n} =$
   (a) $\sqrt{3}$ (b) $\sqrt{\frac{5}{3}}$ (c) $\sqrt{\frac{7}{3}}$ (d) 3

INTEGER MATCH

9. If $\sum_{i=0}^{n}\sum_{j=1}^{n} {}^nC_j \, {}^jC_i = m^n - 1$, then the value of m^2 is

MATCHING LIST

10.

	Column-I		Column-II
P.	If $\frac{\log_e a}{b-c} = \frac{\log_e b}{c-a} = \frac{\log_e c}{a-b}$, then $a^{b+c} \cdot b^{c+a} \cdot c^{a+b} =$	1.	$\frac{37}{29}$
Q.	In an equilateral triangle, 3 coins of radii 1 are kept so that they touch each other and also the sides of the triangle. The area of the triangle is	2.	2
R.	The remainder when $8^{2n} - (62)^{2n+1}$ is divided by 9 is	3.	1
S.	If $\int e^{-2x}(3\cos 5x - 4\sin 5x)dx = e^{-2x}(a\cos 5x + b\sin 5x) + c$, then $a + b =$	4.	$6+4\sqrt{3}$

	P	Q	R	S
(a)	3	4	1	2
(b)	3	4	2	1
(c)	4	3	1	2
(d)	4	1	2	3

See Solution set of Maths Musing 156 on page no. 78

PRACTICE PAPER 2016
JEE MAIN

1. Let $f(\theta) = \begin{vmatrix} \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}$

And the period of $f(\theta)$ is T, then the value of $\frac{4}{\pi} \times T$ is
(a) 2 (b) 3
(c) 4 (d) none of these

2. Let $A = \{x : x$ is a prime factor of 240$\}$
$B = \{x : x$ is the sum of any two prime factor of 240$\}$.
Then $A \cup B$ is
(a) $\{2, 3, 5\}$ (b) $\{2, 3, 5, 7, 8\}$
(c) $\{2, 3, 5, 7\}$ (d) none of these

3. If x_1 and x_2 are the roots of the equation $4^x - 3(2^{x+3}) + 128 = 0$, then the value of $\frac{x_1 + x_2}{|x_1 - x_2|}$ is
(a) 5 (b) 6
(c) 7 (d) 8

4. The number of values of 't' for which there exist a non-zero 3×3 matrix B such that $B' = tB$ is
(a) 0 (b) 1
(c) 2 (d) infinite

5. Let T_n be the number of possible quadrilaterals formed by joining vertices of an n sided regular polygon. If $T_{n+1} - T_n = 20$, then the number of triangles formed by joining the vertices of the polygon is
(a) 15 (b) 16
(c) 20 (d) none of these

6. Coefficient of x^2 in the expansion of

$$E = (x^{2/3} + 4x^{1/3} + 4)^5 \left(\frac{1}{x^{1/3} - 1} + \frac{1}{x^{2/3} + x^{1/3} + 1} \right)^{-9} \text{ is}$$

(a) 20 (b) 24
(c) -18 (d) none of these

7. If $t_1, t_2, t_3, \ldots\ldots, t_n, \ldots\ldots$ are in A.P. such that $t_4 - t_7 + t_{10} = p$, then sum of the first 13 terms of the A.P. is
(a) $10p$ (b) $12p$
(c) $13p$ (d) $15p$

8. If $(20)^{19} + 2(21)(20)^{18} + 3(21)^2(20)^{17} + \ldots\ldots + 20(21)^{19} = m(20)^{19}$, then the value of $\sqrt{m+41}$ is
(a) 20 (b) 21
(c) 22 (d) none of these

9. If $\lim\limits_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = b$, then value of $2b$ is
(a) 1 (b) 3
(c) 4 (d) none of these

10. If $y = \ln \cos\left(\tan^{-1} \frac{e^x - e^{-x}}{2}\right)$

Statement I : $y'(0) = 0$

Statement II : $y'(x) = \frac{e^x - e^{-x}}{1+x^2}$

(a) Statement I is True, Statement II is True; Statement II is a correct explanation for Statement I.
(b) Statement I is True, Statement II is True; Statement II is not a correct explanation for Statement I.
(c) Statement I is True, Statement II is False.
(d) Statement I is False, Statement II is True.

11. The length of the rectangle of the greatest area, that can be inscribed in the ellipse $x^2 + 2y^2 = 8$, are given by
(a) $2\sqrt{6}$ (b) $4\sqrt{6}$
(c) $3\sqrt{6}$ (d) none of these

12. If the primitive of $\frac{1}{(e^x - 1)^2}$ is $f(x) - \ln|g(x)| + C$ then $f(x)$ and $g(x)$ are respectively
(a) $(1 - e^x)$ and $1 - e^{-x}$ (b) $(1 - e^x)^{-1}$ and $1 - e^x$
(c) $(1 - e^x)^{-1}$ and $1 - e^{-x}$ (d) none of these

Contributed by : Sanjay Singh, Mathematics Classes, Chandigarh

13. If $\int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)} = \frac{\pi}{k^2+11}$, then the value of k is

(a) ± 7 (b) ± 8

(c) ± 9 (d) none of these

14. The solution of $\frac{dy}{dx} = \frac{1}{2x-y^2}$ is given by $x = cg(y) + ay^2 + by + d$ where c, a, b, d are constants then $a + b + d$ is

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$

(c) $\frac{5}{4}$ (d) none of these

15. If a, b, c are in A.P., a, x, b are in G.P. and b, y, c are in G.P., then the locus of the point (x, y) is

(a) a straight line (b) a circle

(c) an ellipse (d) a hyperbola

16. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect at two distinct points P and Q, then the line $5x + by - a = 0$ passes through P and Q for

(a) infinitely many values of a

(b) exactly two values of a

(c) exactly one value of a

(d) no value of a

17. AB is a chord of the parabola $y^2 = 4ax$ with the end A at the vertex of the given parabola. BC is drawn perpendicular to AB meeting the axis of the parabola at C. The projection of BC on the axis is

(a) $\frac{1}{4}$(latus-rectum) (b) semi latus-rectum

(c) latus-rectum (d) twice latus-rectum

18. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$.

Statement I: C_1 and C_2 touch each other exactly at two points.

Statement II: Equation of the tangent at (1, 2) to C_1 and C_2 both is $x - y + 1 = 0$ and at (1, –2) is $x + y + 1 = 0$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

19. Let L_1 be the line $2x - 2y + 3z - 2 = 0 = x - y + z + 1$ and L_2 be the line $x + 2y - z - 3 = 0 = 3x - y + 2z - 1$. The vector along the normal of the plane containing the lines L_1 and L_2 is

(a) $7\hat{i} - 7\hat{j} + 8\hat{k}$ (b) $7\hat{i} + 7\hat{j} + 8\hat{k}$

(c) $7\hat{i} - 7\hat{j} + 7\hat{k}$ (d) none of these

20. If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ and $\vec{\delta}$ are unit vectors, then

$|\vec{\alpha}-\vec{\beta}|^2 + |\vec{\beta}-\vec{\gamma}|^2 + |\vec{\gamma}-\vec{\delta}|^2 + |\vec{\delta}-\vec{\alpha}|^2 + |\vec{\gamma}-\vec{\alpha}|^2 + |\vec{\beta}-\vec{\delta}|^2$

does not exceed

(a) 4 (b) 12

(c) 8 (d) 16

21. The distance between the line $\frac{x-2}{1} = \frac{y+2}{-1} = \frac{z-3}{4}$ and the plane $x + 5y + z = 5$ is

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$

(c) $\frac{10}{9}$ (d) $\frac{10}{3\sqrt{3}}$

22. If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then the standard deviation of $x_1, x_2, \ldots, x_{18}$ is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$

(c) $\frac{3}{2}$ (d) none of these

23. Let $A = \{x, y, z, u\}$ and $B = \{a, b\}$. A function $f : A \to B$ is selected randomly. Probability that function is an onto function is

(a) $\frac{1}{8}$ (b) $\frac{5}{8}$

(c) $\frac{7}{8}$ (d) $\frac{3}{8}$

24. If letters of the word "ASSASSIN" are written down at random in a row, the probability that no two S's occur together is

(a) $\frac{1}{17}$ (b) $\frac{1}{14}$

(c) $\frac{1}{28}$ (d) $\frac{1}{35}$

25. If $\cot \alpha$ equals the integral solution of the inequality $4t^2 - 16t + 15 < 0$ and $\sin\beta$ equals to the slope of the bisector of the second quadrant, then

$\sin(\alpha + \beta)\sin(\alpha - \beta)$ is equal to

(a) $-\frac{3}{5}$ (b) $-\frac{4}{5}$

(c) $\frac{2}{\sqrt{5}}$ (d) 3

26. If $\sin^{-1}\left(\frac{a}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then the value of $a^2 + 1$ is

(a) 10 (b) 9 (c) 11 (d) 26

27. The value of $\sin^{-1}(\sin 10)$ is

(a) 10 (b) $3\pi - 10$

(c) $10 - 3\pi$ (d) none of these

28. Statement I : $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Statement II : $\sim(p \leftrightarrow \sim q)$ is a tautology.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

29. Statement I : $\lim_{x\to 0}\left[\frac{\tan^{-1} x}{x}\right]=0$, where $[x]$ represents greatest integer $\le x$.

Statement II : $\frac{\tan^{-1} x}{x} < 1$ for all $x \ne 0$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

30. The natural numbers are divided into rows as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ \ldots & & & & \end{array}$$

Statement I : Sum of the numbers in the 10^{th} row is a number which can be written as sum of two cubes in two different ways.

Statement II : Sum of the numbers in the r^{th} row is $\frac{1}{3}(r^3 + (r+1)^3)$ if $r \ge 2$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true, Statement II is false.

(d) Statement I is false, Statement II is true.

Solutions

1. (a) : $f(\theta)=\begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}$

$(R_1 \to R_1 - R_3;\ R_2 \to R_2 - R_3)$

$$=\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta+4\sin 4\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}(C_1 \to C_1 + C_3)$$

$= -(\cos^2\theta + 1 + \sin^2\theta + 4\sin 4\theta) = -2(1 + 2\sin 4\theta)$

which is periodic function with period $\frac{2\pi}{4}=\frac{\pi}{2}$

Hence the value of $\frac{4}{\pi}\times T=\frac{4}{\pi}.\frac{\pi}{2}=2$

2. (b) : $240 = 2^4 \times 3 \times 5$

So, $A = \{2, 3, 5\}$, $B = \{5, 7, 8\}$

Clearly, $A \cup B = \{2, 3, 5, 7, 8\}$

3. (c) : Put $2^x = y$, given equation becomes

$y^2 - 3(8y) + 128 = 0 \Rightarrow y^2 - 24y + 128 = 0$

$\Rightarrow (y - 8)(y - 16) = 0 \Rightarrow y = 16, 8$

$\Rightarrow 2^x = 16, 8 \Rightarrow x = 4, 3$

Hence $\frac{x_1+x_2}{|x_1-x_2|}=\frac{4+3}{|4-3|}=7$

4. (c) : We have, $B = (B')' = (tB)' = tB' = t^2B$

As $B \ne 0$, $t^2 = 1 \Rightarrow t = \pm 1$

5. (c) : T_n = Number of ways of choosing four vertices out of $n = {}^nC_4$

$\therefore\ 20 = T_{n+1} - T_n = {}^{n+1}C_4 - {}^nC_4$

$= {}^nC_4 + {}^nC_3 - {}^nC_4 = {}^nC_3$

$[\because\ {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$\therefore\ 20 = \frac{n(n-1)(n-2)}{3.2.1}$

$\Rightarrow n(n-1)(n-2) = 120 = 6.5.4$

$\Rightarrow n = 6$

$\therefore$ Required number of triangles $= {}^nC_3 = {}^6C_3 = 20$.

6. (c) : $E = ((x^{1/3} + 2)^2)^5\left(\frac{x^{2/3}+x^{1/3}+1+x^{1/3}-1}{x-1}\right)^{-9}$

$$=(2+x^{1/3})^{10}\left(\frac{x^{2/3}+2x^{1/3}}{x-1}\right)^{-9}$$

$$=(2+x^{1/3})^{10}\frac{(x-1)^9}{(x^{1/3})^9(x^{1/3}+2)^9} = (2+x^{1/3})x^{-3}(x-1)^9$$

Coefficient of x^2

$= 2$[coefficient of x^5 in the expansion of $x^{-3}(x-1)^9$]

$= 2(-1)^9$ [coefficient of x^8 in $(1-x)^9$]

$= 2(-1)\,{}^9C_8\,(-1)^8 = -18$

7. (c) : Let d be the common difference of the A.P., then

$p = t_4 - t_7 + t_{10} = t_4 + (t_{10} - t_7)$

$\quad = t_1 + 3d + 3d = t_1 + 6d$

Now, $S_{13} = \frac{13}{2}\,[2t_1 + (13-1)d] = 13p$

8. (b) : Dividing the given equation by $(20)^{19}$, we get $m = 1 + 2x + 3x^2 + \ldots\ldots + 20x^{19}$,

where $x = \frac{21}{20}$.

$$m=\frac{d}{dx}[x+x^2+\ldots\ldots+x^{20}]=\frac{d}{dx}\left[\frac{x(x^{20}-1)}{x-1}\right]$$

$$=\frac{(x-1)(21x^{20}-1)-x(x^{20}-1)}{(x-1)^2}=\frac{20x^{21}-21x^{20}+1}{(x-1)^2}$$

But $20x^{21}-21x^{20}=20\left(\frac{21}{20}\right)^{21}-21\left(\frac{21}{20}\right)^{20}=0$

$$\therefore\ m=\frac{1}{\left(\frac{21}{20}-1\right)^2}=400$$

Hence, $\sqrt{m+41}=\sqrt{441}=21$

9. (b) : $b=\lim\limits_{x\to 0}\dfrac{e^{x^2}-1+1-\cos x}{x^2}$

$$=\lim_{x\to 0}\left(\frac{e^{x^2}-1}{x^2}+\frac{2\sin^2 x/2}{4(x/2)^2}\right)=1+\frac{1}{2}=\frac{3}{2}$$

Hence $2b = 2(3/2) = 3$

10. (c) : $y'(x)=-\dfrac{\sin\left(\tan^{-1}\dfrac{e^x-e^{-x}}{2}\right)}{\cos\left(\tan^{-1}\dfrac{e^x-e^{-x}}{2}\right)}$

$$\times\frac{1}{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\times\frac{e^x+e^{-x}}{2}$$

$$=-\tan\left(\tan^{-1}\frac{e^x-e^{-x}}{2}\right)\times\frac{2}{(e^x+e^{-x})^2}\times(e^x+e^{-x})$$

$$=-\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$y'(0) = 0$

11. (a) : Any point on the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$ is $(2\sqrt{2}\cos\theta,\ 2\sin\theta)$.

A = Area of the inscribed rectangle

$= 4(2\sqrt{2}\cos\theta)\,(2\sin\theta)=8\sqrt{2}\sin 2\theta$

$$\frac{dA}{d\theta}=16\sqrt{2}\cos 2\theta=0\ \Rightarrow\ \theta=\pi/4$$

Also $\dfrac{d^2A}{d\theta^2}=-32\sqrt{2}\sin 2\theta<0$ for $\theta=\dfrac{\pi}{4}$.

Hence, the inscribed rectangle is of largest area if the sides are

$4\sqrt{2}\cos\frac{\pi}{4}$ and $4\sin\left(\frac{\pi}{4}\right)$ *i.e.*, 4 and $2\sqrt{2}$.

Hence the length of diagonal

$$=\sqrt{4^2+(2\sqrt{2})^2}=\sqrt{16+8}=\sqrt{24}=2\sqrt{6}$$

12. (c) : $\displaystyle\int\frac{1}{(e^x-1)^2}dx=\int\frac{e^x\,dx}{e^x(e^x-1)^2}=\int\frac{dt}{t(t-1)^2}$

$$=\int\frac{1}{t-1}\left[\frac{1}{t-1}-\frac{1}{t}\right]dt=\int\left(\frac{1}{(t-1)^2}-\frac{1}{t-1}+\frac{1}{t}\right)dt$$

$=\dfrac{1}{1-t} - \ln|t-1| + \ln|t| + C$

$\qquad = (1-e^x)^{-1} - \ln|1-e^{-x}| + C$

Hence, $f(x) = (1-e^x)^{-1}$ and $g(x) = 1 - e^{-x}$

13. (a) : $\displaystyle\int_0^\infty\frac{dx}{(x^2+4)(x^2+9)}=\frac{1}{5}\int_0^\infty\left[\frac{1}{x^2+4}-\frac{1}{x^2+9}\right]dx$

$$=\frac{1}{5}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}-\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^\infty$$

$$=\frac{1}{5}\left[\frac{\pi}{4}-\frac{\pi}{6}\right]=\frac{\pi}{60},\ \text{so } k^2+11=60$$

$\Rightarrow k^2 = 49 \Rightarrow k = \pm 7$

14. (c) : We have, $\dfrac{dx}{dy} = 2x - y^2$

This is a linear equation in x.

The integrating factor is $e^{-\int 2dy}=e^{-2y}$

So $\dfrac{d}{dy}(xe^{-2y})=-y^2e^{-2y}$

Integrating, we have

$$xe^{-2y} = \frac{-y^2e^{-2y}}{-2} - \int ye^{-2y}dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{ye^{-2y}}{2} - \frac{1}{2}\int e^{-2y}dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{y}{2}e^{-2y} + \frac{e^{-2y}}{4} + C$$

$$\therefore\ x = Ce^{2y} + \frac{y^2}{2} + \frac{y}{2} + \frac{1}{4}$$

On comparing with the above equation, we get

$$a = \frac{1}{2}, b = \frac{1}{2}, d = \frac{1}{4}$$

Hence $a + b + d = \frac{5}{4}$.

15. (b) : We have $2b = a + c$, $x^2 = ab$, $y^2 = bc$ so that $x^2 + y^2 = b(a + c) = 2b^2$, which is a circle.

16. (d) : Equation of the line passing through the points P and Q is

$x^2 + y^2 + 2ax + cy + a - (x^2 + y^2 - 3ax + dy - 1) = 0$

$\Rightarrow\ 5ax + (c - d)y + a + 1 = 0$

which coincides with $5x + by - a = 0$

if $\frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}$

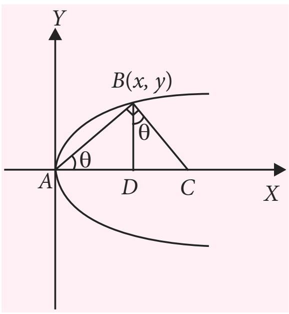
or if $a^2 + a + 1 = 0$ which does not give any real value of a.

17. (c) : Draw BD perpendicular to the axis of the parabola. Let the coordinates of B be (x, y) then slope of AB is given by $\tan\theta = \frac{y}{x}$

Projection of BC on the axis of the parabola is

$DC = BD\tan\theta$

$$= y\left(\frac{y}{x}\right) = \frac{y^2}{x} = \frac{4ax}{x} = 4a$$

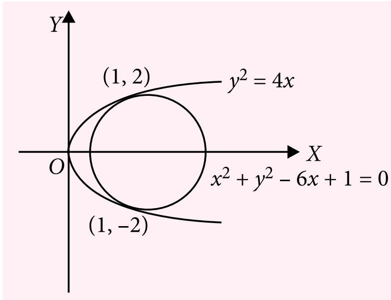


18. (a) : Solving for the points of intersection, we have

$x^2 + 4x - 6x + 1 = 0$

$\Rightarrow\ (x - 1)^2 = 0$

$\Rightarrow\ x = 1 \ \Rightarrow\ y = \pm 2$



Thus the two curves meet at (1, 2) and (1, –2).

Tangent at (1, 2) to $y^2 = 4x$ is

$y(2) = 2(x + 1) \Rightarrow\ x - y + 1 = 0$

Tangent at (1, 2) to the circle C_2 is

$x + 2y - 3(x + 1) + 1 = 0$

or $x - y + 1 = 0$ same as the tangent to the curve C_1.

Similarly the tangent at the point (1, –2) to the two curves is $x + y + 1 = 0$

$\Rightarrow$ Statement II is true and hence statement I is also true.

19. (a) : The plane through L_1 is

$2x - 2y + 3z - 2 + \lambda(x - y + z + 1) = 0$

and the plane through L_2 is

$x + 2y - z - 3 + \mu(3x - y + 2z - 1) = 0$

Two planes are same

So $\frac{2+\lambda}{1+3\mu} = \frac{2+\lambda}{\mu-2} = \frac{3+\lambda}{2\mu-1} = \frac{2-\lambda}{3+\mu}$

$\Rightarrow\ \mu = -\frac{3}{2}$ and $\lambda = 5$ and the equation of the plane is $7x - 7y + 8z + 3 = 0$.

Hence the vector along the normal is $7\hat{i} - 7\hat{j} + 8\hat{k}$.

20. (d) : We have

$$0 \le |\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}|^2 = |\vec{\alpha}|^2 + |\vec{\beta}|^2 + |\vec{\gamma}|^2 + |\vec{\delta}|^2$$
$$+ 2(\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha} + \vec{\alpha}\cdot\vec{\delta} + \vec{\gamma}\cdot\vec{\delta} + \vec{\beta}\cdot\vec{\delta})$$

$$= 4 + 2(\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha} + \vec{\alpha}\cdot\vec{\delta} + \vec{\gamma}\cdot\vec{\delta} + \vec{\beta}\cdot\vec{\delta})$$

So, $\vec{\alpha}\cdot\vec{\beta} + \vec{\beta}\cdot\vec{\gamma} + \vec{\gamma}\cdot\vec{\alpha} + \vec{\alpha}\cdot\vec{\delta} + \vec{\gamma}\cdot\vec{\delta} + \vec{\beta}\cdot\vec{\delta} \ge -2$

Now $|\vec{\alpha}-\vec{\beta}|^2 + |\vec{\beta}-\vec{\gamma}|^2 + |\vec{\gamma}-\vec{\delta}|^2 + |\vec{\delta}-\vec{\alpha}|^2$
$+ |\vec{\gamma}-\vec{\alpha}|^2 + |\vec{\beta}-\vec{\delta}|^2$

$$= 3(|\vec{\alpha}|^2 + |\vec{\beta}|^2 + |\vec{\gamma}|^2 + |\vec{\delta}|^2)$$
$$-2(\vec{\alpha}\cdot\vec{\beta} + \vec{\gamma}\cdot\vec{\delta} + \vec{\delta}\cdot\vec{\alpha} + \vec{\alpha}\cdot\vec{\gamma} + \vec{\beta}\cdot\vec{\delta} + \vec{\beta}\cdot\vec{\gamma}) \le 3(4) - 2(-2) = 16$$

21. (d) : Direction ratios of the line are 1, –1, 4 and that of the normal to the plane are 1, 5, 1.

Since $1 \times 1 + (-1)(5) + 4 \times 1 = 0$

$\therefore$ Normal to the plane is perpendicular to the line and thus the line is parallel to the plane and the distance

between them is the distance of any point, let (2, –2, 3) on the line from the plane and is equal to

$$\left|\frac{2(1)+(-2)(5)+3\times1-5}{\sqrt{1+25+1}}\right| = \frac{10}{\sqrt{27}} = \frac{10}{3\sqrt{3}}$$

22. (c) : Let $d_i = x_i - 8$ but

$$\sigma_x^2 = \sigma_d^2 = \frac{1}{18}\sum d_i^2 - \left(\frac{1}{18}\sum d_i\right)^2$$

$$= \frac{1}{18}\times 45 - \left(\frac{9}{18}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$$

Therefore $\sigma_x = 3/2$

23. (c) : Total number of functions from A to B is $2^4 = 16$. There are exactly two functions which are not onto viz. one in which all the elements of A are mapped to a and other in which all the elements of A are mapped to b. Thus, there are 14 onto functions.

Therefore probability of required event is $\frac{14}{16} = \frac{7}{8}$

24. (b) : The number of ways of permuting the letters of the word "ASSASSIN" is

$$\frac{8!}{2!4!} = \frac{8\times7\times6\times5}{2} = 840$$

To find the number of ways is which two S are not together,

We first arrange A, A, I, N. This can be done in $\frac{4!}{2!}$ = 12 ways.

If in the following figure X positions of show one such arrangement, then 4S's can be arranged at four boxes out of the five boxes.

$$\square \times \square \times \square \times \square \times \square$$

Thus, corresponding to each arrangement of A, A, I, N, there are ${}^5C_4 = 5$ ways of arranging S's.

$\therefore$ The number of favourable ways = (12) × (5) = 60

Thus, required probability = $\frac{60}{840} = \frac{1}{14}$

25. (b) : We have, $4t^2 - 16t + 15 < 0$

$$\Rightarrow \frac{3}{2} < t < \frac{5}{2}$$

$\Rightarrow$ cot α = 2, the integral solution of the given inequality and sin β = tan 135° = –1

$\therefore$ $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

$$= \frac{1}{1+\cot^2\alpha} - 1 = \frac{1}{1+4} - 1 = -\frac{4}{5}$$

26. (a) : $\sin^{-1}\left(\frac{a}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}\left(\frac{a}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{a}{5}\right) + \cos^{-1}\sqrt{1-\frac{16}{25}} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{a}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$$

$\Rightarrow a = 3$ as $\sin^{-1} a + \cos^{-1} a = \frac{\pi}{2}\ \forall\ a$

Hence $a^2 + 1 = 10$

27. (b) : Let $y = \sin^{-1}(\sin 10)$

$\Rightarrow \sin y = \sin 10 = \sin(3\pi + (10 - 3\pi))$

$= -\sin(10 - 3\pi)$

$= \sin(3\pi - 10)$

$\Rightarrow y = 3\pi - 10$

28. (c) : Table for basic logical connectives.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$(p \leftrightarrow q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

Note that $\sim(p \leftrightarrow \sim q)$ is not a tautology.

Statement II is false.

And $\sim(p \leftrightarrow \sim q)$ is equivalent to $(p \leftrightarrow q)$

Thus, statement I is true.

29. (c) : Let $f(x) = \tan^{-1} x - x$

$$f'(x) = \frac{1}{1+x^2} - 1 = \frac{-x^2}{1+x^2} < 0$$

Hence f is a decreasing function, so for $x > 0$,

$\tan^{-1}x < x \Rightarrow \frac{\tan^{-1}x}{x} < 1$ and for $x < 0$, $f(x) > f(0) = 0$

$$\Rightarrow \tan^{-1}x > x \Rightarrow \frac{\tan^{-1}x}{x} < 1\ (x < 0)$$

Thus $\lim_{x\to0^+}\left[\frac{\tan^{-1}x}{x}\right] = 0$ and $\lim_{x\to0^-}\left[\frac{\tan^{-1}x}{x}\right] = 0$

30. (c) : The r^{th} row contains $(2r - 1)$ numbers and its last term is r^2.

When read from right to left, the r^{th} row forms an A.P. with first term r^2 and common difference – 1.

$\therefore$ Sum of the terms in the r^{th} row

$$= \frac{2r-1}{2}[2(r^2) + (2r-1-1)(-1)]$$

$= (2r - 1)(r^2 - r + 1) = r^3 + (r - 1)^3$

Sum of the terms in the 10^{th} row

$= 10^3 + 9^3 = 1729 = 12^3 + 1^3$

VITEEE-2016 B.Tech Engineering Entrance Exam Application forms sales begins

VIT University Chancellor Dr. G. Viswanathan is seen inaugurating the sale of the VITEEE-2016 application forms, to some of the aspiring students for the tests, in April 2016 at the Vellore Head Post Office on Friday. Seen others in the picture are Superintendent of Posts A. Natarajan, Deputy Superintendent R. Mahendran, VIT University's VC Dr. Anand A Samuel, Pro-VC Dr. S. Narayan, VPs Sankar Viswanathan, Sekar Viswanathan and G.V. Selvam, Director UG Admissions, K. Manivannan, PRO- (Posts) Kathir Ahmed and Marketing Executive S. Selvakumar.

The sale of application forms for the VIT University Entrance Examinations (VITEEE-2016) to be held in April 2016, for B.Tech courses various streams, began in all the 92 Head Post Offices, with the VIT University Chancellor **Dr. G. Viswanathan** inaugurating it at the Head Post Office, here, on Friday.

It is scheduled that the entrance examinations for the B.Tech offered in the VIT University, Vellore and Chennai Campuses, will be held from **April 6th to April 17th 2016**. This Computer Based Test (CBT) is held in **118 cities including Dubai, Kuwait and Muscat**.

The University offers courses in Vellore Campus - Bio-Medical Engineering, Biotechnology, Computer Science and Engineering (Specialisation in Bioinformatics), Civil Engineering, Chemical Engineering, Computer Science and Engineering, Electronics and Communication Engineering, Electrical and Electronics Engineering, Electronics and Instrumentation Engineering, Information Technology, Mechanical Engineering, Mechanical Engineering (Specialisation in Automotive Engineering), Mechanical Engineering (Specialisation in Energy Engineering), Production and Industrial Engineering and in **Chennai Campus** – B.Tech in Civil Engineering, Computer Science and Engineering, Electronics and Communication Engineering, Electrical and Electronics Engineering and Mechanical Engineering.

The entrance examination application forms from the Head Post Offices, it can be obtained by sending a Demand Draft for **Rs.990/-** drawn in favour of VIT University, payable at Vellore to the Director – UG Admissions or by cash payment at selected post offices across the country. Issuing of online and offline application has commenced from **November 27th 2015**.

Candidates can also apply online at **www.vit.ac.in** (online applicants need to pay **Rs.960/-** only). The last date for applying is **29th February, 2016**.

VIT University has been consistently been ranked among the premier engineering institutions of the country by India Today. VIT has an impressive track record of placement. For the 8th year in a row, VIT has achieved the top slot in placements.

Visit: **www.vit.ac.in** for further details.

JEE WORK OUTS

1. Let (x_0, y_0) be the solution of the following equations $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$. Then x_0 is

(a) 1/6 (b) 1/3 (c) 1/2 (d) 6

2. Let $S(K) = 1 + 3 + 5 + ... + (2K - 1) = 3 + K^2$. Then which of the following is true?

(a) $S(1)$ is correct

(b) Principle of mathematical induction can be used to prove the formula

(c) $S(K) \nRightarrow S(K + 1)$

(d) $S(K) \Rightarrow S(K + 1)$

3. The real part of $(1 - \cos\theta + 2i\sin\theta)^{-1}$ is

(a) $\dfrac{1}{3+5\cos\theta}$ (b) $\dfrac{1}{5-3\cos\theta}$

(c) $\dfrac{1}{3-5\cos\theta}$ (d) $\dfrac{1}{5+3\cos\theta}$

4. a, g, h are arithmetic mean, geometric mean and harmonic mean between two positive numbers x and y respectively. Then identify the correct statement among the following

(a) h is the harmonic mean between a and g

(b) No such relation exists between a, g and h

(c) g is the geometric mean between a and h

(d) A is the arithmetic mean between g and h

5. If $s_r = \alpha^r + \beta^r + \gamma^r$, then the value of $\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$ is equal to

(a) 0 (b) $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$

(c) $(\alpha + \beta + \gamma)^6$ (d) $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2$

6. p : Sita is beautiful; q = Sita is clever.
When $\sim p \vee q$ is written in verbal form it becomes

(a) Sita is not beautiful or Sita is not clever

(b) Sita is beautiful and she is clever

(c) Sita is not beautiful or Sita is clever

(d) Sita is beautiful and Sita is not clever

7. Ten persons, amongst whom are A, B and C to speak at a function. The number of ways in which it can be done if A wants to speak before B and B wants to speak before C is

(a) $\dfrac{10!}{6}$ (b) 3! 7!

(c) ${}^{10}P_3 \cdot 7!$ (d) none of these

8. If the coefficient of x^7 in $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \dfrac{1}{bx^2}\right)^{11}$, then ab =

(a) 1 (b) 1/2 (c) 2 (d) 3

9. If the probability density function of a random variable X is $f(x) = x/2$ in $0 \le x \le 2$, then $P(X > 1.5 \mid x > 1)$ is equal to

(a) 7/16 (b) 3/4 (c) 7/12 (d) 21/64

10. The product of perpendiculars drawn from the origin to the lines represented by the equation, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, will be

(a) $\dfrac{ab}{\sqrt{a^2 - b^2 + 4h^2}}$ (b) $\dfrac{bc}{\sqrt{a^2 - b^2 + 4h^2}}$

(c) $\dfrac{ca}{\sqrt{(a^2 + b^2) + 4h^2}}$ (d) $\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}$

11. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, (where $pq \neq 0$) are bisected by the x-axis, then

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(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

12. A circle C_1 of radius 2 touches both x-axis and y-axis. Another circle C_2 whose radius is greater than 2 touches circle C_1 and both the axes. Then the radius of circle C_2 is

(a) $6-4\sqrt{2}$ (b) $6+4\sqrt{2}$
(c) $6-4\sqrt{3}$ (d) $6+4\sqrt{3}$

13. If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then $\sin\alpha + \cos\alpha$ and $\sin\alpha - \cos\alpha$ must be equal to

(a) $\sqrt{2}\cos\theta, \sqrt{2}\sin\theta$ (b) $\sqrt{2}\sin\theta, \sqrt{2}\cos\theta$
(c) $\sqrt{2}\sin\theta, \sqrt{2}\sin\theta$ (d) $\sqrt{2}\cos\theta, \sqrt{2}\cos\theta$

14. A pair of straight lines drawn through the origin form with the line $2x + 3y = 6$ an isosceles right angled triangle, then the area of the triangle thus formed is

(a) $\Delta = \frac{36}{13}$ (b) $\Delta = \frac{12}{17}$
(c) $\Delta = \frac{13}{5}$ (d) none of these

15. Point D, E are taken on the side BC of a triangle ABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} =$

(a) 1 (b) 2
(c) 4 (d) none of these

16. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z$ is equal to

(a) xyz (b) 0 (c) 1 (d) $2xyz$

17. From the bottom of a pole of height h, the angle of elevation of top of a tower is α and the pole subtends angle β at the top of the tower. The height of the tower is

(a) $\frac{h\tan(\alpha-\beta)}{\tan(\alpha-\beta)-\tan\alpha}$ (b) $\frac{h\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$
(c) $\frac{\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$ (d) none of these

18. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are

(a) $\frac{-3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}$

(b) $\frac{-5+\sqrt{3}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$

(c) $\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}, \frac{5+\sqrt{3}}{2}$

(d) $-3-\sqrt{5}, -3+\sqrt{5}, 3-\sqrt{5}, 3+\sqrt{5}$

19. The value of $\lim_{x\to\pi/2} \frac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)}$ is

(a) ∞ (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/8$

20. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$ then $f^2(4) + g^2(4)$ is

(a) 8 (b) 16 (c) 32 (d) 64

21. The value of p for which the function
$$f(x) = \begin{cases} \dfrac{(4^x-1)^3}{\sin\dfrac{x}{p}\log_e\left(1+\dfrac{x^2}{3}\right)}, & x \neq 0 \\ 12(\log_e 4)^3, & x = 0 \end{cases}$$
is continuous at $x = 0$, is

(a) 1 (b) 2
(c) 3 (d) none of these

22. Maximize $z = 3x + y$ subject to $x + y \le 6$, $y \le 4$, $x \ge 1$, $x \ge 0$ and $y \ge 0$.

(a) 20 (b) 18 (c) 10 (d) 15

23. Let $f(x) = \int_0^x \frac{\cos t}{t}dt, x > 0$ then $f(x)$ has

(a) maxima when $n = -2, -4, -6, ...$
(b) maxima when $n = -1, -3, -5, ...$
(c) maxima when $n = 1, 3, 5, ...$
(d) none of these

24. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}$, $4\hat{i}$, $3\hat{i}+3\hat{j}$ and $-3\hat{i}+2\hat{j}$ respectively. The quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither a rhombus nor a rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

25. Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)\}$ be a relation in A. Then R is

(a) reflexive and transitive
(b) reflexive and symmetric
(c) transitive and symmetric
(d) equivalence

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Series 8

ACE YOUR WAY CBSE XI

STATISTICS AND PROBABILITY

STATISTICS

INTRODUCTION

- Statistics deals with data collected for specific purposes. Statistical analysis involves the process of collecting data, analyzing data and then summarizing it into numerical form.
- Measures of central tendency are not sufficient to give complete information about a given data.
- Variability is another factor which is required to be studied under statistics. Like 'measures of central tendency' we want to have a single number to describe variability. This single number is called a 'measure of dispersion'.

MEASURES OF DISPERSION

It measures the degree of scatteredness of the observations in the given data around the central value.

There are following measures of dispersion:

(i) Range (ii) Quartile deviation

(iii) Mean deviation (iv) Standard deviation

RANGE

It is the difference between two extreme observations of the given distribution (arranged in ascending or descending order).

i.e., Range = Maximum value – Minimum value

MEAN DEVIATION

Mean deviation of a distribution is the arithmetic mean of the absolute deviations of the terms of the distribution from its statistical mean (A.M., median or mode).

The sum of deviations from mean ($\bar{x}$) is zero.

Mean deviation for ungrouped data

Let $x_1, x_2, x_3, \ldots, x_n$ be the given n observations. Let $\bar{x}$ be the mean and M be the median. Then,

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

$$MD(M) = \frac{\sum_{i=1}^{n} |x_i - M|}{n}$$

Mean deviation for grouped data

(1) For discrete frequency distribution :

$$MD(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

$$MD(M) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|, \text{ where } N = \sum_{i=1}^{n} f_i$$

(2) For continuous frequency distribution :

$$MD(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

$$MD(M) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|$$

Here, x_i's are the mid-points of the corresponding classes

Shortcut Method for finding mean deviation

(1) About mean – For this, find the mean by step-deviation method, *i.e.*,

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{N} \times h, \text{ where } d_i = \frac{x_i - a}{h}$$ and a is the assumed mean.

And then applying the formula,

$$MD(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

(2) About median – For this, find the median by the formula,

$$M = l + \frac{\frac{N}{2} - C}{f} \times h$$

where l, C, f and h are the lower limit, the cumulative frequency of the class just preceding the median class, the frequency and the width of the median class respectively.
And then

$$MD(M) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|$$

Limitations of mean deviation

(1) The mean deviation about median calculated for such series (where the degree of variability is very high) can not be fully relied.

(2) The mean deviation about the mean is not very scientific since the sum of the deviations from the mean (if we ignored the minus signs) is more than the sum of the deviations from median.

(3) The mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment.

VARIANCE AND STANDARD DEVIATION

The variance of a variate is the mean of the squares of all deviations of the values of the variate x from the arithmetic mean of the observations and is denoted by σ^2 or var (x).
Thus, if x_1, x_2, ..., x_n be n given values of the variate and $\bar{x}$ be their mean, then

Variance, $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

The dispersion about mean of a set of observations is expressed as the positive square-root of the variance and is called the standard deviation which is denoted by σ and S.D. is given by

For ungrouped data	$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
For a discrete frequency distribution	$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$
For continuous frequency distribution	$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$ or $\sigma = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$

Shortcut method to find variance and standard deviation

- $\sigma^2 = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i y_i^2 - \left\{\frac{1}{N}\sum_{i=1}^{n} f_i y_i\right\}^2\right]$
- $\sigma = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$

where h and N are the width of the class-intervals and the sum of frequencies respectively and $y_i = \frac{x_i - a}{h}$, a = assumed mean

ANALYSIS OF FREQUENCY DISTRIBUTIONS

The measure of variability which is independent of units is called coefficient of variation and denoted as C.V.
The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100, \bar{x} \neq 0,$$

Where σ and $\bar{x}$ are the standard deviation and mean of the data respectively. For two series with equal means, the series with greater standard deviation (or variance) is called more variable or dispersed than the other. Also, the series with lesser value of standard deviation (or variance) is said to be more consistent than the other.
The series having greater C.V. is said to be more variable than the other. The series having lesser C.V. is said to be more consistent than the other.

Note: (1) The mean of two groups of observations taken together is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

(2) The variance of two groups of observations taken together is

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\sigma_1^2 + n_2\sigma_2^2 + \left(\frac{n_1 n_2}{n_1 + n_2}\right)(\bar{x}_1 - \bar{x}_2)^2\right]$$

where, x_1, x_2 are the respective means of the two groups having n_1 and n_2 observations.

PROBABILITY

DEFINITION

Probability is a concept which numerically measures the degree of uncertainty and, therefore, of certainty of the occurrence of an event.

EXPERIMENT

An operation which can produce some well-defined outcomes is known as an experiment.

RANDOM EXPERIMENT

An experiment whose outcome cannot be predicted with certainty is called a random experiment. An experiment whose outcome can be foretold before hand is not a random experiment.

OUTCOMES AND SAMPLE SPACE

- **Outcome :** A possible result of a random experiment is called its outcome.
- **Sample space :** The set of all possible outcomes of a random experiment is called the sample space for that experiment. It is usually denoted by S.
- **Sample point :** Each element of the sample space is called a sample point.

EVENT

A subset of the sample space S is called an event.

- **Occurrence of an event :** For a random experiment, let E be an event and let $E = \{a, b, c\}$. If the outcome of the experiment is a or b or c, then we say that event E has occurred.

Types of events

(1)	**Impossible and Sure events :** The empty set ϕ is also a subset of sample space S and it represents an impossible event. Sample space S is also a subset of S, it represents a sure event.
(2)	**Simple Event :** An event is said to be a simple event if it is a singleton subset of the sample space S.
(3)	**Compound Event :** An event which is not a simple event is called compound event. In other words, an event is said to be compound event if it has more than one sample point.

ALGEBRA OF EVENTS

(1)	**Complementary Event :** Corresponding to every event A, we define an event "not A" which is said to occur when A does not occur. Thus, the event "not A" denoted by A' or $\bar{A}$ is called the complementary event of A.
(2)	**The Event 'A or B' :** Let A and B be two events associated with a sample space, then the event 'A or B' denoted by $A \cup B$ is the event 'either A or B or both' $A \cup B = \{x : x : \in A \text{ or } x \in B\}$
(3)	**The Event 'A and B' :** If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'. $A \cap B = \{x : x \in A \text{ and } x \in B\}$
(4)	**The Event 'A but not B' :** $A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A - B$ denotes the event 'A but not B'. We know that $A - B = A \cap B'$

MUTUALLY EXCLUSIVE EVENTS

Two or more events are said to be mutually exclusive if one of them occurs, others cannot occur. Thus two or more events are said to be mutually exclusive if no two of them can occur together.

Thus events $A_1, A_2, ..., A_n$ are mutually exclusive if and only if $A_i \cap A_j = \phi$ for $i \neq j$

EXHAUSTIVE EVENTS

For a random experiment, a set of events (cases) is said to be exhaustive if atleast one of them must necessarily happen every time the experiment is performed.

Note : If $E_i \cap E_j = \phi$ for $i \neq j$ *i.e.*, events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called mutually exclusive and exhaustive events.

Remarks :

(1) For any event E, $P(E) \geq 0$.

(2) $P(S) = 1$

(3) Let S be a sample space and E be an event, such that $n(S) = l$ and $n(E) = m$. If each outcome is equally likely, then it follows that

$$P(E) = \frac{m}{l} = \frac{\text{Number of outcomes favourable to } E}{\text{Total possible outcomes}}$$

(4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are disjoint sets, *i.e.*, they are mutually exclusive events, then $A \cap B = \phi$.

Therefore $P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have $P(A \cup B) = P(A) + P(B)$

(5) $P(A') = P(\text{not } A) = 1 - P(A)$

Very Short Answer Type

1. If standard deviation of n numbers $x_1, x_2, \ldots, x_n$ is a, then write the value of standard deviation of numbers $2x_1, 2x_2, \ldots\ldots, 2x_n$.

2. Two coins are tossed once. Find the probability of getting 1 head and 1 tail.

3. Find the mean deviation about the mean for the following data :

$$15, 17, 10, 13, 10$$

4. What is the probability that a number selected from the numbers 1, 2, 3,..., 25, is prime number when each of the given numbers is equally likely to be selected?

5. If $\frac{3}{10}$ is the probability that an event will happen, what is the probability that it will not happen?

Short Answer Type

6. Find the mean deviation about the median of the following frequency distribution:

Class	0-6	6-12	12-18	18-24	24-30
Frequency	8	10	12	9	5

7. The probability that at least one of the events E_1 and E_2 occurs is 0.6. If the probability of the simultaneous occurrence of E_1 and E_2 is 0.2, find $P(\bar{E}_1) + P(\bar{E}_2)$.

8. An urn contains 9 red, 7 white and 4 black balls. If two balls are drawn at random, find the probability that:
(i) both the balls are red
(ii) one ball is white

9. The sum and the sum of squares of length x (in cm) and weight y (in g) of 30 plant products are given below:

$$\sum_{i=1}^{30} x_i = 112, \sum_{i=1}^{30} x_i^2 = 502.4, \sum_{i=1}^{30} y_i = 194 \text{ and } \sum_{i=1}^{30} y_i^2 = 1275.4$$

Which is more variable, the length or weight?

10. If a leap year is selected at random, what is the chance that it will contain 53 Tuesdays?

Long Answer Type

11. Find the mean deviation about the mean for the following data:

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	8	4	2	12	2	4	8

12. Find the probability that when a hand of 7 cards is dealt from a well-shuffled deck of 52 cards, it contains:
(i) all 4 kings (ii) exactly 3 kings
(iii) at least 3 kings

13. Calculate mean, variance and standard deviation for the following frequency distribution.

Class	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	2	4	9	10	9	4	2

14. A natural number is chosen at random from among the first 500 natural numbers. What is the probability that the number so chosen is divisible by 3 or 5?

15. Suppose that samples of polythene bags from two manufactures, A and B are tested by a prospective buyer for bursting pressure, with the following results:

Bursting pressure (in kg)	Number of bags manufactured by manufacturer	
	A	B
5-10	2	9
10-15	9	11
15-20	29	18
20-25	54	32
25-30	11	27
30-35	5	13

Which set of bag has the highest average bursting pressure? Which has more uniform pressure?

SOLUTIONS

1. If the standard deviation of n numbers $x_1, x_2, \ldots, x_n$ is a. Then the standard deviation of numbers $2x_1, 2x_2, \ldots\ldots, 2x_n$ is $2a$.

2. When two coins are tossed once, the sample space is given by $S = \{HH, HT, TH, TT\}$ and therefore, $n(S) = 4$.
Let E = event of getting 1 head and 1 tail. Then, $E = \{HT, TH\}$ and therefore, $n(E) = 2$.

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}.$$

3. Let the mean of the given data be $\bar{x}$. Then,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{65}{5} = 13 \qquad [\because \ n = 5]$$

The values of $|x_i - \bar{x}|$ are 2, 4, 3, 0, 3

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} = \frac{12}{5} = 2.4$$

Hence, $MD(\bar{x}) = 2.4$

4. Let S be the sample space associated with the given experiment and A be the event "selecting a prime number". Then, $S = \{1, 2, 3,..., 25\}$ and
$A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

$\therefore$ Total number of elementary events = 25

Favourable number of elementary events = 9

Hence, required probability $= \frac{9}{25}$.

5. Let E be the event. Then,

$$P(E) = \frac{3}{10} \Rightarrow P(\bar{E}) = \{1 - P(E)\} = \left(1 - \frac{3}{10}\right) = \frac{7}{10}.$$

Hence, $P(\text{not } E) = \frac{7}{10}$.

6. Calculation of Mean Deviation about the Median

Class	Mid values (x_i)	Frequency (f_i)	Cumulative Frequency ($c.f.$)	$\lvert x_i - 14\rvert$	$f_i\lvert x_i - 14\rvert$
0-6	3	8	8	11	88
6-12	9	10	18	5	50
12-18	15	12	30	1	12
18-24	21	9	39	7	63
24-30	27	5	44	13	65
		$N = \Sigma f_i$		$\Sigma f_i\lvert x_i - 14\rvert = 278$	

Here $N = 44$, so $\frac{N}{2} = 22$ and the cumulative frequency just greater than $\frac{N}{2}$ is 30. Thus 12-18 is the median class.

Now, median $= l + \frac{N/2 - C}{f} \times h$,

where $l = 12$, $h = 6$, $f = 12$, $C = 18$

So, median $= 12 + \frac{22-18}{12} \times 6 = 12 + \frac{4 \times 6}{12} = 14$

$\therefore$ Mean deviation about median

$$= \frac{1}{N}\sum_{i=1}^{5} f_i |x_i - 14| = \frac{278}{44} = 6.318$$

7. Given, $P(E_1 \cup E_2) = 0.6$ and $P(E_1 \cap E_2) = 0.2$

$\therefore$ $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow$ $P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$
$= (0.6 + 0.2) = 0.8$

$\Rightarrow$ $P(E_1) + P(E_2) = 0.8$

$\Rightarrow$ $\{1 - P(\bar{E}_1)\} + \{1 - P(\bar{E}_2)\} = 0.8$

$\Rightarrow$ $P(\bar{E}_1) + P(\bar{E}_2) = (2 - 0.8) = 1.2$

Hence, $P(\bar{E}_1) + P(\bar{E}_2) = 1.2$

8. There are 20 balls in the bag out of which 2 balls can be drawn in ${}^{20}C_2$ ways. So, total number of elementary events $= {}^{20}C_2 = 190$

(i) There are 9 red balls out of which 2 balls can be drawn in ${}^{9}C_2$ ways.

$\therefore$ Favourable number of elementary events
$= {}^{9}C_2 = 36$

So, required probability $= \frac{36}{190} = \frac{18}{95}$.

(ii) There are 7 white balls out of which one ball can be drawn in ${}^{7}C_1$ ways. One ball from the remaining 13 balls can be drawn in ${}^{13}C_1$ ways. Therefore, one white and one ball of other colour can be drawn in ${}^{7}C_1 \times {}^{13}C_1$ ways.

So, favourable number of elementary events
$= {}^{7}C_1 \times {}^{13}C_1$.

Hence, required probability $= \frac{{}^{7}C_1 \times {}^{13}C_1}{{}^{20}C_2} = \frac{91}{190}$.

9. We have, $\text{Var}(x) = \frac{1}{30} \cdot \sum_{i=1}^{30} x_i^2 - \left(\frac{1}{30} \cdot \sum_{i=1}^{30} x_i\right)^2$

$$= \left(\frac{502.4}{30}\right) - \left(\frac{112}{30}\right)^2 = \frac{251.2}{15} - \left(\frac{56}{15}\right)^2$$

$$= \frac{251.2}{15} - \frac{3136}{225} = \frac{3768 - 3136}{225} = \frac{632}{225} = 2.8$$

$$\text{Var}(y) = \frac{1}{30} \cdot \sum_{i=1}^{30} y_i^2 - \left(\frac{1}{30} \cdot \sum_{i=1}^{30} y_i\right)^2 = \left\{\frac{1275.4}{30} - \left(\frac{194}{30}\right)^2\right\}$$

$$= \frac{637.7}{15} - \left(\frac{97}{15}\right)^2$$

$$= \frac{637.7}{15} - \frac{9409}{225} = \frac{9565.5 - 9409}{225} = \frac{156.5}{225} = 0.6956$$

$\therefore$ $\text{Var}(x) > \text{Var}(y)$.

$\therefore$ Length is more variable than weight.

10. A leap year has 366 days *i.e.*, 52 complete weeks and two days more. These two days will be the two consecutive days of a week. A leap year will have 53 Tuesday if out of the two consecutive days of a week selected at random, one is a Tuesday.

Let S be the sample space and E be the event that out of the two consecutive days of a week, one is Tuesday.

S = {(Mon, Tues), (Tues, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat), (Sat, Sun), (Sun, Mon)}

$\therefore$ $n(S) = 7$

and E = {(Mon, Tues), (Tues, Wed)}

$\therefore$ $n(E) = 2$

Required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{2}{7}$

11. We will find the mean $(\bar{x})$, let us compute $\bar{x}$ by step-deviation method. The formula for the same is

$$\bar{x} = a + h\left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)$$

where $d_i = \dfrac{x_i - a}{h}$, a = assumed mean and h = width of class

Let us take assumed mean $a = 45$ and $h = 10$ and form the following table:

Marks obtained	Number of students (f_i)	Mid-points (x_i)	$d_i = \dfrac{x_i - 45}{10}$	$f_i d_i$	$\lvert x_i - \bar{x}\rvert = \lvert x_i - 45\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
10-20	8	15	–3	–24	30	240
20-30	4	25	–2	–8	20	80
30-40	2	35	–1	–2	10	20
40-50	12	45	0	0	0	0
50-60	2	55	1	2	10	20
60-70	4	65	2	8	20	80
70-80	8	75	3	24	30	240
	$N = 40$			$\Sigma f_i d_i = 0$		$\Sigma f_i \lvert x_i - \bar{x}\rvert = 680$

$$\therefore\ \bar{x} = a + h\left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)$$

$$\Rightarrow\ \bar{x} = 45 + 10 \times \frac{0}{40} = 45$$

$$\therefore\ MD(\bar{x}) = \frac{1}{N}\Sigma f_i \lvert x_i - \bar{x}\rvert = \frac{680}{40} = 17$$

12. From a deck of 52 playing cards, 7 cards can be drawn in ${}^{52}C_7$ ways.

$\therefore$ Total number of elementary events = ${}^{52}C_7$

(i) There are 4 kings. Therefore, 4 kings out of 4 kings and 3 other cards from the remaining 48 cards can be chosen in ${}^{4}C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_4 \times {}^{48}C_3$

Hence, required probability $= \dfrac{{}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Three kings out of 4 kings and 4 other cards out of remaining 48 cards can be chosen in ${}^{4}C_3 \times {}^{48}C_4$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_3 \times {}^{48}C_4$

Hence, required probability $= \dfrac{{}^{4}C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) When 7 cards are drawn from a deck of 52 playing cards, then getting at least 3 kings means: getting 3 kings and 4 other cards or getting 4 kings and 3 other cards. This can be done in

${}^{4}C_3 \times {}^{48}C_4 + {}^{4}C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_3 \times {}^{48}C_4 + {}^{4}C_4 \times {}^{48}C_3$

Hence, required probability

$$= \frac{{}^{4}C_3 \times {}^{48}C_4 + {}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \frac{46}{7735}$$

13. Here, $h = 30$. Let the assumed mean be $a = 105$.

Now, we prepare the table given below.

Class	Frequency (f_i)	Mid-point (x_i)	$y_i = \dfrac{(x_i - 105)}{30}$	y_i^2	$f_i y_i$	$f_i y_i^2$
0-30	2	15	–3	9	–6	18
30-60	4	45	–2	4	–8	16
60-90	9	75	–1	1	–9	9
90-120	10	105 = a	0	0	0	0
120-150	9	135	1	1	9	9
150-180	4	165	2	4	8	16
180-210	2	195	3	9	6	18
Total	$N = 40$				0	86

$\therefore$ $a = 105$, $h = 30$, $N = \Sigma f_i = 40$, $\Sigma f_i y_i = 0$ and $\Sigma f_i y_i^2 = 86$

$$\therefore\ \bar{x} = \left(a + \frac{\Sigma f_i y_i}{N} \times h\right)$$

$$\Rightarrow\ \bar{x} = \left(105 + \frac{0}{30} \times 30\right) = 105$$

Thus, mean = 105.

Variance, $\sigma^2 = \frac{h^2}{N^2} \cdot \{N \cdot \Sigma f_i y_i^2 - (\Sigma f_i y_i)^2\}$

$$= \frac{(30)^2}{(40)^2} \cdot \{40 \times 86 - (0)^2\} = 1935$$

Standard deviation, $\sigma = \sqrt{1935} = 43.98$.

14. Let S be the sample space. Then, clearly $n(S) = 500$

Let E_1 = event of getting a number divisible by 3, and E_2 = event of getting a number divisible by 5. Then, $(E_1 \cap E_2)$ = event of getting a number divisible by both 3 and 5 = event of getting a number divisible by 15.

$\therefore$ $E_1 = \{3, 6, 9, ..., 495, 498\}$,

$E_2 = \{5, 10, 15, ..., 495, 500\}$

and $(E_1 \cap E_2) = \{15, 30, 45, ..., 495\}$

$$\therefore\ n(E_1) = \left(\frac{498}{3}\right) = 166,\ n(E_2) = \left(\frac{500}{5}\right) = 100$$

and $n(E_1 \cap E_2) = \left(\frac{495}{15}\right) = 33$

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{166}{500} = \frac{83}{250},$$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{100}{500} = \frac{1}{5}$$

and $P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{33}{500}$

$\therefore$ P(The chosen number is divisible by 3 or 5)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \left(\frac{83}{250} + \frac{1}{5} - \frac{33}{500}\right) = \frac{233}{500}$$

Hence, the required probability is $\frac{233}{500}$.

15. For determining the set of bags having higher average bursting pressure, we compute mean and for finding out set of bags having more uniform pressure we compute coefficient of variation.

Manufacture A:

Computation of mean and standard deviation

Bursting pressure	Mid-values (x_i)	Frequency (f_i)	$u_i = \frac{x_i - 17.5}{5}$	u_i^2	$f_i u_i$	$f_i u_i^2$
5-10	7.5	2	–2	4	–4	8
10-15	12.5	9	–1	1	–9	9
15-20	17.5 = a	29	0	0	0	0
20-25	22.5	54	1	1	54	54
25-30	27.5	11	2	4	22	44
30-35	32.5	5	3	9	15	45
		N = 110	$\Sigma u_i = 3$		$\Sigma f_i u_i = 78$	$\Sigma f_i u_i^2 = 160$

$$\bar{x}_A = a + h\left(\frac{\sum_{i=1}^{n} f_i u_i}{N}\right)$$

$$\Rightarrow\ \bar{x}_A = 17.5 + 5 \times \frac{78}{110} \qquad [\because h = 5,\ a = 17.5]$$

$= 21.04$

$$\sigma_A^2 = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]$$

$$\Rightarrow\ \sigma_A^2 = 25\left[\frac{160}{110} - \left(\frac{78}{110}\right)^2\right]$$

$$\Rightarrow\ \sigma_A^2 = 25\left(\frac{17600 - 6084}{110 \times 110}\right) = 23.79$$

$$\Rightarrow\ \sigma_A = \sqrt{23.79} = 4.87$$

$\therefore$ Coefficient of variation

$$= \frac{\sigma_A}{\bar{x}_A} \times 100 = \frac{4.87}{21.04} \times 100 = 23.14$$

Manufacture B:

Bursting pressure	Mid-values x_i	frequency (f_i)	$u_i = \frac{x_i - 17.5}{5}$	u_i^2	$f_i u_i$	$f_i u_i^2$
5-10	7.5	9	−2	4	−18	36
10-15	12.5	11	−1	1	−11	11
15-20	17.5 = a	18	0	0	0	0
20-25	22.5	32	1	1	32	32
25-30	27.5	27	2	4	54	108
30-35	32.5	13	3	9	39	117
		N = 110	$\Sigma u_i = 3$		$\Sigma f_i u_i = 96$	$\Sigma f_i u_i^2 = 304$

$$\bar{x}_B = a + h\left(\frac{\sum_{i=1}^{n} f_i u_i}{N}\right)$$

$$\Rightarrow \bar{x}_B = 17.5 + 5 \times \frac{96}{110} = 17.5 + 4.36 = 21.86$$

$$\sigma_B^2 = h^2\left[\frac{1}{N}\left(\sum_{i=1}^{n} f_i u_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]$$

$$\Rightarrow \sigma_B^2 = 25\left[\frac{304}{110} - \left(\frac{96}{110}\right)^2\right]$$

$$\Rightarrow \sigma_B^2 = 25\left(\frac{33440 - 9216}{110 \times 110}\right) = 50.04$$

$$\Rightarrow \sigma_B = \sqrt{50.04} = 7.07$$

$\therefore$ Coefficient of variation

$$= \frac{\sigma_B}{\bar{x}_B} \times 100 = \frac{7.07}{21.86} \times 100 = 32.34$$

We observe that the average bursting pressure is higher for manufacture B. So, bags manufactured by B have higher bursting pressure.

The coefficient of variation is less for manufacture A. So, bags manufactured by A have more uniform pressure.

JEE WORK CUTS

Contd. from Page No. 22

26. The value of a so that the volume of parallelopiped formed by $\hat{i} + a\hat{j} + a\hat{k}$, $\hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum is

(a) −3 (b) 3 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

27. $\int \frac{dx}{(1+x^2)\sqrt{p^2 + q^2(\tan^{-1} x)^2}} =$

(a) $\frac{1}{q}\log\left[q\tan^{-1} x + \sqrt{p^2 + q^2(\tan^{-1} x)^2}\right] + c$

(b) $\log\left[q\tan^{-1} x + \sqrt{p^2 + q^2(\tan^{-1} x)^2}\right] + c$

(c) $\frac{2}{3q}(p^2 + q^2 \tan^{-1} x)^{3/2} + c$

(d) none of these

28. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70, 80 were wrongly read as 40, 20, 50 respectively. The correct mean is

(a) 80 (b) 50 (c) $82\frac{1}{2}$ (d) 48

29. A tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the coordinate axes in P and Q. The least value of PQ is

(a) $a - b$ (b) $a + b$

(c) $\frac{a+b}{2}$ (d) $2a$

30. The plane passing through the point (−2, −2, 2) and containing the line joining the points (1, 1, 1) and (1, −1, 2) makes intercepts on the co-ordinate axes, the sum whose length is

(a) 3 (b) 4 (c) 6 (d) 12

Answer keys

1.	(c)	2.	(d)	3.	(d)	4.	(c)	5.	(d)
6.	(c)	7.	(a)	8.	(a)	9.	(c)	10.	(d)
11.	(d)	12.	(b)	13.	(a)	14.	(d)	15.	(c)
16.	(a)	17.	(b)	18.	(a)	19.	(c)	20.	(c)
21.	(d)	22.	(b)	23.	(b)	24.	(a)	25.	(b)
26.	(c)	27.	(a)	28.	(b)	29.	(b)	30.	(d)

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CONCEPT BOOSTERS

Class XI

CIRCLES

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This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CIRCLES

A circle is the locus of a point which moves in such a way that its distance from a fixed point is constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

EQUATION OF THE CIRCLE IN VARIOUS FORMS

- The simplest equation of the circle is $x^2 + y^2 = r^2$, whose centre is (0, 0) and radius is r.
- The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle with centre (a, b) and radius r.
- The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is the general equation of a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Diametric Form

- Equation of the circle with points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Equation of a Circle under different conditions

Condition	Equation
Touches both the axes with radius a	$(x - a)^2 + (y - a)^2 = a^2$
Touches x-axis only with centre (a, b)	$(x - a)^2 + (y - b)^2 = a^2$
Touches y-axis only with centre (a, b)	$(x - a)^2 + (y - b)^2 = a^2$

Parametric Equation of a Circle

- The equation $x = a\cos\theta$, $y = a\sin\theta$ are called parametric equations of the circle $x^2 + y^2 = a^2$ and θ is called a parameter. The point $(a\cos\theta, a\sin\theta)$ is also referred to as point θ. The parametric coordinates of any point on the circle $(x - h)^2 + (y - k)^2 = a^2$ are given by $(h + a\cos\theta, k + a\sin\theta)$ with $0 \le \theta < 2\pi$.

Remarks

- Since there are three independent constants g, f, c in the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, a circle can be found to satisfy three independent geometrical conditions and no more. For example, when three points on a circle or three tangents to a circle or two tangents to a circle and a point on it are given, the circle can be determined.
- To find the condition for the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent the circle, viz., $x^2 + y^2 + 2gx + 2fy + c = 0$, we see that there is no term in xy and that the coefficient of x^2 is same as that of y^2 *i.e.*, the coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.
- The equation of a straight line joining two points α and β on the circle $x^2 + y^2 = a^2$ is $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$.

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Intercepts made on axes

Solving the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with $y = 0$ we get, $x^2 + 2gx + c = 0$.

If discriminant $4(g^2 - c)$ is positive, *i.e.*, if $g^2 > c$, the circle will meet the x-axis at two distinct points, say $(x_1, 0)$ and $(x_2, 0)$ where $x_1 + x_2 = -2g$ and $x_1x_2 = c$.

The intercept made on x-axis by the circle

$$= |x_1 - x_2| = 2\sqrt{g^2 - c}.$$

In the similar manner if $f^2 > c$, intercept made on y-axis $= 2\sqrt{f^2 - c}$.

- If $g^2 - c > 0 \Rightarrow$ circle cuts x-axis at two distinct points.
- If $g^2 = c \Rightarrow$ circle touches the x-axis
- If $g^2 < c \Rightarrow$ circle lies completely above or below the x-axis.

Case is similar for intercepts on y-axis.

TANGENTS & NORMALS

A tangent to a curve at a point is defined as the limiting positions of a secant obtained by joining the given point to another point in the vicinity on the curve as the second point tends to the first point along the curve.

OR

As the limiting position of a secant obtained by joining two points on the curve in the vicinity of the given point as both the points tend to the given point.

Two tangents, real or imaginary, can be drawn to a circle from a point in the plane. The tangents are real and distinct if the point is outside the circle, real and coincident if the point is on the circle, and imaginary if the point is inside the circle.

Equation of Tangents

- If $S = 0$ be a curve than $S_1 = 0$ indicate the equation which is obtained by substituting $x = x_1$ and $y = y_1$ in the equation of the given curve and $T = 0$ is the equation which is obtained by substituting $x^2 = xx_1$, $y^2 = yy_1$, $2xy = xy_1 + yx_1$, $2x = x + x_1$, $2y = y + y_1$ in the equation $S = 0$.
- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
  Equation of the tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$ at $A(x_1, y_1)$ is
  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
  In particular, equation of the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$

Normals

The normal to a curve at a point is defined as the straight line passing through the point and perpendicular to the tangent at that point. In case of a circle, every normal passes through the centre of the circle.

Equation of Normal

- The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) lying on the circle is $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$.
  In particular, equation of the normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

Equation of Tangent in Slope Form

- The condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$ and the point of contact is $(-a^2m/c, a^2/c)$ *i.e.*, $y = mx \pm a\sqrt{1 + m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$ whatever be the value of m.

Equation of Tangent in Parametric Form

Since parametric co-ordinates of circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$, then equation of tangent at $(a\cos\theta, a\sin\theta)$ is

$x \cdot a\cos\theta + y \cdot a\sin\theta = a^2$ or $x\cos\theta + y\sin\theta = a$

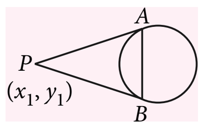
LENGTH OF A TANGENT AND POWER OF A POINT

The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$.

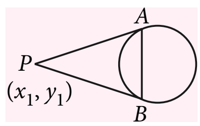
Square of length of the tangent from the point P is also called the power of a point w.r.t. a circle.

Note : Power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

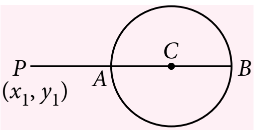
- **Chord of contact :** If two tangents PA and PB are drawn from an outside point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact AB is
  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$



- **Equation of Pair of Tangents :** The joint equation of a pair of tangents drawn from the point $P(x_1, y_1)$ to the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ is $T^2 = SS_1$.



- **Director Circle:** The locus of the point of intersection of perpendicular tangents is called director circle. If $(x - \alpha)^2 + (y - \beta)^2 = r^2$ is the equation of a circle then its director circle is $(x - \alpha)^2 + (y - \beta)^2 = 2r^2$.
- **Equation of the Chord with a given Middle Point:** The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ with its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$.
  This on simplification can be put in the form
  $$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
  which is designated by $T = S_1$.
  The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.
  The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.
- **Position of a Point with respect to a Circle:** The point $P(x_1, y_1)$ lies outside, on, or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >= \text{ or } < 0$.
  The greatest and the least distance of a point P from a circle with centre C and radius r is $PC + r$ and $PC - r$ respectively.



Position of the Line with respect to the Circle

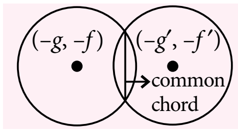
- If p is perpendicular from the centre on the line, then
  - $p = r \Leftrightarrow$ the line touches the circle or the line is a tangent of the circle.
  - $0 < p < r \Leftrightarrow$ the line is a secant of the circle.
  - $p = 0 \Leftrightarrow$ the line is a diameter of the circle.
  - Chord of contact exists only if the point 'P' is outside the circle.
    If R is radius of the circle and L is the length of the tangent from (x_1, y_1) on the circle $S = 0$, then
  - Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$
  - Area of the triangle formed by the pair of tangents and its chord of contact $= \frac{RL^3}{R^2 + L^2}$
  - Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\frac{2RL}{L^2 - R^2}\right)$
  - Equation of the circle circumscribing the triangle PT_1T_2 is
    $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

RADICAL AXIS

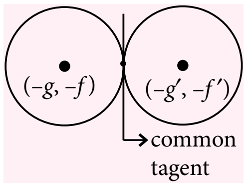
The radical axis of two circles is the locus of a point from which the tangent segments to the two circles are of equal length.

Equation of the Radical Axis

- In general $S - S' = 0$ represents the equation of the radical axis to the two circles.
  i.e., $2x(g - g') + 2y(f - f') + c - c' = 0$
  where $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$
  and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$
- If $S = 0$ and $S' = 0$ intersect in real and distinct points then $S - S' = 0$ is the equation of the common chord of the two circles.



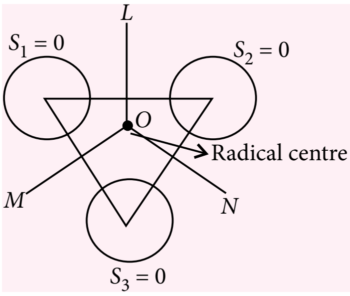
- If $S = 0$ and $S' = 0$ touch each other, then $S - S' = 0$ is the equation of the common tangent to the two circles at the point of contact.



- Radical axis is always perpendicular to the line joining the centres of the two circles.
- Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.
  Radical axis bisects a common tangent between the two circles.
- A system of circles, every two of which have the same radical axis, is called a coaxial system.
  Pairs of circles which do not have radical axis are concentric.

Radical Centre

- The radical axes of three circles, taken in pairs, are concurrent.
  Let the equations of the three circles S_1, S_2 and S_3 be
  $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$
  $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
  and $S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ ...(1)



Now, by the previous section, the radical axis of S_1 and S_2 is obtained by subtracting the equations of these circles ; hence it is

$$S_1 - S_2 = 0 \qquad ...(2)$$

Similarly, the radical axis of S_2 and S_3 is

$$S_2 - S_3 = 0 \qquad ...(3)$$

The lines (2) and (3) meet at a point whose coordinates say, (X, Y) satisfy

$$S_1 - S_2 = 0 \text{ and } S_2 - S_3 = 0;$$

hence the coordinates (X, Y) satisfy

$$(S_1 - S_2) + (S_2 - S_3) = 0;$$

that is, $S_1 - S_3 = 0 \qquad ...(4)$

But (4) is the radical axis of the circles S_1 and S_3 and hence the three radical axes are concurrent. The point of concurrency of the three radical axes is called the radical centre.

FAMILY OF CIRCLES

- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are two intersecting circles, then $S + lS' = 0$, $l \neq -1$, is the equation of the family of circles passing through the points of intersection of $S = 0$ and $S' = 0$.
- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle which is intersected by the straight line $\mu \equiv ax + by + c = 0$ at two real and distinct points, then $S + l\mu = 0$ is the equation of the family of circles passing through the points of intersection of $S = 0$ and $\mu = 0$.
- If $\mu = 0$ touches $S = 0$ at P, then $S + l\mu = 0$ is the equation of the family of circles, each touching $\mu = 0$ at P.
- The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form.

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+l\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

where l is a parameter.

- The equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any value of m is $(x - x_1)^2 + (y - y_1)^2 + l(y - y_1 - m(x - x_1)) = 0$.
- Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.
- Equation of circles circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ where value of λ can be found out by using condition that coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

THE CONDITION THAT TWO CIRCLES SHOULD INTERSECT

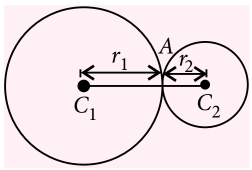
A necessary and sufficient condition for the two circles to intersect at two distinct points is

$r_1 + r_2 > C_1C_2 > |r_1 - r_2|$,

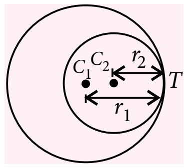
where C_1, C_2 be the centres and r_1, r_2 be the radii of the two circles.

External and Internal Contacts of Circles

If two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively, touch each other externally, $C_1C_2 = r_1 + r_2$. Coordinates of the point of contact are



$$A \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$

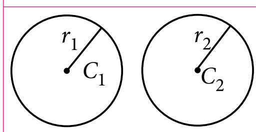
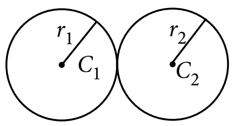


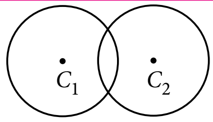
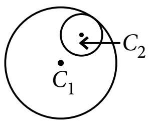
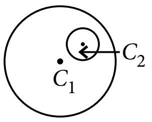
The circles touch each other internally if $C_1C_2 = r_1 - r_2$.

Coordinates of the point of contact are

$$T \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

- **Common Tangents to Two Circles**

	Number of tangents	Condition
	4 common tangents (2 direct and 2 transverse)	$r_1 + r_2 < C_1C_2$
	3 common tangents	$r_1 + r_2 = C_1C_2$

	2 common tangents	$\lvert r_1 - r_2\rvert < C_1C_2 < r_1 + r_2$
	1 common tangent	$\lvert r_1 - r_2\rvert = C_1C_2$
	No common tangent	$C_1C_2 < \lvert r_1 + r_2\rvert$

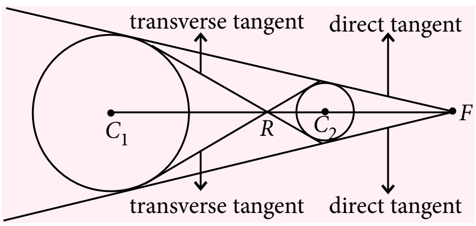
Length of an External (or Direct) Common Tangent & Internal (or Transverse) Common Tangent

Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles is given by :

$$L_{\text{ext}} = \sqrt{d^2 - (r_1 - r_2)^2} \quad \text{and} \quad L_{\text{int}} = \sqrt{d^2 - (r_1 + r_2)^2}$$

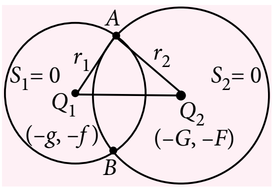
where d = distance between the centres of the two circles, r_1 and r_2 are the radii of the two circles.

- The length of internal common tangent is always less than the length of the external or direct common tangent.
- The direct common tangents to two circles meet on the line of centres and divide it externally in the ratio of the radii.
- The transverse common tangents also meet on the line of centres and divide it internally in the ratio of the radii.



ORTHOGONAL CIRCLES

Two circles are said to be orthogonal if the tangents to the circles at either point of intersection are at right angles. In fig. Q_1 and Q_2 are the centres of the circles



$$S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and $S_2 \equiv x^2 + y^2 + 2Gx + 2Fy + C = 0$

The circles, S_1 and S_2, intersect at A and B.

The tangent at A to the circle S_1 is perpendicular to the radius Q_1A and the tangent at A to S_2 is perpendicular to the radius Q_2A. Hence, if the two tangents are at right angles, then the radii Q_1A and Q_2A must also be at right angles. Accordingly, the condition that S_1 and S_2 should be orthogonal is that $\angle Q_1AQ_2$ should be 90°; by Pythagoras' theorem, this condition is equivalent to

$$Q_1Q_2^2 = Q_1A^2 + Q_2A^2 = r_1^2 + r_2^2$$

$$\Rightarrow (g - G)^2 + (f - F)^2 = g^2 + f^2 - c + G^2 + F^2 - C$$

or, on simplification, $2(gG + fF) = c + C$

Since, AQ_2 is perpendicular to the radius Q_1A, the tangent at A to the circle S_1 passes through the centre of the circle S_2; similarly, the tangent at A to S_2 passes through the centre of S_1.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. Equation of circle touching the line $|x - 2| + |y - 3| = 4$ will be

(a) $(x - 2)^2 + (y - 3)^2 = 12$
(b) $(x - 2)^2 + (y - 3)^2 = 4$
(c) $(x - 2)^2 + (y - 3)^2 = 10$
(d) $(x - 2)^2 + (y - 3)^2 = 8$

2. The equation of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is

(a) $(\sqrt{2}-1)a$ or $(\sqrt{2}+1)a$
(b) $\sqrt{2}a$ or $2\sqrt{2}a$
(c) $(2-\sqrt{2})a$ or $(2+\sqrt{2})a$
(d) None of these

3. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

4. C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_3 which does not pass through C_2 is

(a) $x - \sqrt{3}y + 2 = 0$ (b) $\sqrt{3}x - y + 2 = 0$
(c) $\sqrt{3}x - y - 2 = 0$ (d) $x + \sqrt{3}y + 2 = 0$

5. Circles are drawn through the point (2, 0) to cut intercept of length '5' units on the x-axis. If their centres lie in the first quadrant then their equation is
(a) $x^2 + y^2 - 9x + 2ky + 14 = 0, k > 0$
(b) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0, k > 0$
(c) $x^2 + y^2 - 9x - 2ky + 14 = 0, k > 0$
(d) $x^2 + y^2 - 2ky - 9y + 14 = 0, k > 0$

6. The equation of the circle passing through the point (2, –1) and having two diameters along the pair of lines $2x^2 + 6y^2 - x + y - 7xy - 1 = 0$ is
(a) $x^2 + y^2 + 10x + 6y - 19 = 0$
(b) $x^2 + y^2 + 10x - 6y + 19 = 0$
(c) $x^2 + y^2 + 10x + 6y + 19 = 0$
(d) $x^2 + y^2 - 10x + 6y + 19 = 0$

7. An equilateral triangle has two vertices (–2, 0) and (2, 0) and its third vertex lies below the x-axis. The equation of the circumcircle of the triangle is
(a) $\sqrt{3}(x^2+y^2)-4y+4\sqrt{3}=0$
(b) $\sqrt{3}(x^2+y^2)-4y-4\sqrt{3}=0$
(c) $\sqrt{3}(x^2+y^2)+4y+4\sqrt{3}=0$
(d) $\sqrt{3}(x^2+y^2)+4y-4\sqrt{3}=0$

8. The coordinates of two points on the circle $x^2 + y^2 - 12x - 16y + 75 = 0$ one nearest to the origin and the other farthest from it, are
(a) (3, 4), (9, 12) (b) (3, 2), (9, 12)
(c) (–3, 4), (9, 12) (d) (3, 4), (9, –12)

9. The centre of a circle of radius $4\sqrt{5}$ lies on the line $y = x$ and satisfies the inequality $3x + 6y > 10$. If the line $x + 2y = 3$ is a tangent to the circle, then the equation of the circle is
(a) $\left(x-\frac{23}{3}\right)^2+\left(y-\frac{23}{3}\right)^2=80$
(b) $\left(x+\frac{17}{3}\right)^2+\left(y+\frac{17}{3}\right)^2=80$
(c) $\left(x+\frac{23}{3}\right)^2+\left(y-\frac{23}{3}\right)^2=80$
(d) $\left(x-\frac{17}{3}\right)^2+\left(y-\frac{17}{3}\right)^2=80$

10. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circles $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal then which of the following equations can represent L_1?
(a) $x + y = 0$ (b) $x - y = 0$
(c) $7x - y = 0$ (d) $x - 7y = 0$

11. A variable circle passes through the fixed point $A(p, q)$ and touches x-axis. The locus of the other end of the diameter through A is
(a) $(y - p)^2 = 4qx$ (b) $(x - q)^2 = 4py$
(c) $(x - p)^2 = 4qy$ (d) $(y - q)^2 = 4px$

12. The chord of contact of tangents from a point 'P' to a circle passes through Q. If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to
(a) $\frac{l_1+l_2}{2}$ (b) $\frac{l_1-l_2}{2}$
(c) $\sqrt{l_1^2+l_2^2}$ (d) $\sqrt{l_1^2-l_2^2}$

13. If the chord of contact of tangents from 3 points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will
(a) be concyclic.
(b) be collinear.
(c) form the vertices of triangle.
(d) none of these.

14. The equation of circumcircle of a ΔABC is $x^2 + y^2 + 3x + y - 6 = 0$. If $A = (1, -2)$, $B = (-3, 2)$ and the vertex C varies then the locus of orthocentre of ΔABC is a
(a) straight line (b) circle
(c) parabola (d) ellipse

15. Point A lies on $y = x$ and point B on $y = mx$ so that length of $AB = 4$ units. Then value of 'm' for which locus of mid point of AB represents a circle is
(a) $m = 0$ (b) $m = -1$
(c) $m = 2$ (d) $m = -2$

16. $x^2 + y^2 + 6x + 8y = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are the equation of the two circles. Equation of one of their common tangent is
(a) $7x - 5y - 1 - 5\sqrt{74} = 0$
(b) $7x - 5y - 1 + 5\sqrt{74} = 0$
(c) $7x - 5y + 1 - 5\sqrt{74} = 0$
(d) $5x - 7y + 1 - 5\sqrt{74} = 0$

17. P and Q are any two points on the circle $x^2 + y^2 = 4$ such that PQ is a diameter. If α and β are the lengths of perpendicular from P and Q on $x + y = 1$ then the maximum value of $\alpha\beta$ is
(a) 1/2 (b) 7/2 (c) 1 (d) 2

18. From a point P outside a circle with centre at C, tangents PA and PB are drawn such that $\frac{1}{(CA)^2}+\frac{1}{(PA)^2}=\frac{1}{16}$, then the length of chord AB is

(a) 8 (b) 12

(c) 16 (d) none of these

19. If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then

(a) $2 < r < 8$ (b) $r < 2$

(c) $r = 2$ (d) $r > 2$

20. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects to first circle at R above the x-axis and the line segment PQ at S. The maximum area of the triangle QSR is

(a) $\frac{2}{9}$ (b) $\frac{5\sqrt{2}}{7}$ (c) $\frac{4\sqrt{3}}{9}$ (d) $\frac{\sqrt{2}}{13}$

21. The locus of the centre of a circle $x^2 + y^2 - 6x - 6y + 14 = 0$ which touches the circle externally and also the y-axis is given by

(a) $x^2 - 6x - 7y + 14 = 0$

(b) $x^2 - 10x - 6y + 14 = 0$

(c) $y^2 - 6x - 10y + 14 = 0$

(d) $y^2 - 10x - 6y + 14 = 0$

SECTION-II

Multiple Correct Answer Type

22. Consider the circle $x^2 + y^2 - 8x - 18y + 93 = 0$ with centre 'C' and point $P(2, 5)$ outside it. From the point P, a pair of tangents PQ and PR are drawn to the circle with S as the midpoint of QR. The line joining P to C intersects the given circle at A and B. Which of the following hold(s) good ?

(a) CP is the arithmetic mean of AP and BP

(b) PR is the geometric mean of PS and PC

(c) PS is the harmonic mean of PA and PB

(d) The angle between the two tangents from P is $\tan^{-1}\left(\frac{3}{4}\right)$

23. Q is any point on the circle $x^2 + y^2 = 9$. QN is perpendicular from Q on the x-axis. Locus of the point of trisection of QN is

(a) $4x^2 + 9y^2 = 36$ (b) $9x^2 + 4y^2 = 36$

(c) $9x^2 + y^2 = 9$ (d) $x^2 + 9y^2 = 9$

24. $(1, 2\sqrt{2})$ is a point on circle $x^2 + y^2 = 9$. Then the points on the given circle which are at 2 unit distance from $(1, 2\sqrt{2})$ is are :

(a) $(-1, 2\sqrt{2})$ (b) $(2\sqrt{2}, 1)$

(c) $\left(\frac{23}{9}, \frac{10\sqrt{2}}{9}\right)$ (d) $\left(\frac{23}{9}, \frac{10}{9}\right)$

25. Let $C_1 = x^2 + y_2 = r_1^2$, $C_2 = x^2 + y^2 = r_2^2$ $(r_1 < r_2)$ be two circles. Let A be the fixed point $(r_1, 0)$ on C_1 and B be a variable point on C_2. Let the line BA meet the circle C_2 again at E. Then,

(a) the maximum value of BE is $2r_2$.

(b) the minimum value of BE is $2\sqrt{r_2^2 - r_1^2}$.

(c) if O is origin, then, the best possible lower bound for $OA^2 + OB^2 + BE^2$ is $5r_2^2 - 3r_1^2$.

(d) if O is origin, then, the best possible upper bound for $OA^2 + OB^2 + BE^2$ is $r_1^2 + 5r_2^2$.

26. If Q, S are two points on the circle $x^2 + y^2 = 4$ such that the tangents QP, SR are parallel. If PS, QR intersect at T then $\left(\frac{QT}{PQ}\right)^2+\left(\frac{ST}{RS}\right)^2+PQ\cdot RS \neq$

(a) 5 (b) 10 (c) 16 (d) 17

27. If A and B are two points on the circle $x^2 + y^2 - 4x + 6y - 3 = 0$, which are farthest and nearest respectively from the point $(7, 2)$ then

(a) $A \equiv (2-2\sqrt{2}, -3-2\sqrt{2})$

(b) $A \equiv (2+2\sqrt{2}, -3+2\sqrt{2})$

(c) $B \equiv (2+2\sqrt{2}, -3+2\sqrt{2})$

(d) $B \equiv (2-2\sqrt{2}, -3-2\sqrt{2})$

28. If two circles $x^2 + y^2 - 6x - 12y + 1 = 0$ and $x^2 + y^2 - 4x - 2y - 11 = 0$ cut a third circle orthogonally then the radical axis of the two circles passes through

(a) (1, 1)

(b) (0, 6)

(c) centre of the third circle.

(d) mid-point of the line joining the centres of the given circles.

29. The equation of a circle C_1 is $x^2 + y^2 = 4$. The locus of the intersection of orthogonal tangents to the circle is the curve C_2 and the locus of the intersection of perpendicular tangents to the curve C_2 is the curve C_3 then

(a) C_3 is a circle.
(b) the area enclosed by the curve C_3 is 8π.
(c) C_2 and C_3 are circles with the same centre.
(d) none of the above.

30. Consider the circles $C_1 \equiv x^2 + y^2 - 2x - 4y - 4 = 0$ and $C_2 \equiv x^2 + y^2 + 2x + 4y + 4 = 0$ and the line $L \equiv x + 2y + 2 = 0$, then
(a) L is the radical axis of C_1 and C_2.
(b) L is the common tangent of C_1 and C_2.
(c) L is the common chord of C_1 and C_2.
(d) L is perpendicular to the line joining centres of C_1 and C_2.

31. A circle touches the line $x + y - 2 = 0$ at (1, 1) and cuts the circle at $x^2 + y^2 + 4x + 5y - 6 = 0$ at P and Q. Then
(a) PQ can never be parallel to the given line $x + y - 2 = 0$.
(b) PQ can never be perpendicular to the given line $x + y - 2 = 0$.
(c) PQ always passes through (6, –4).
(d) PQ always passes through (–6, 4).

32. If $al^2 - bm^2 + 2dl + 1 = 0$, where a, b, d are fixed real numbers such that $a + b = d^2$ then the line $lx + my + 1 = 0$ touches a fixed circle. Then the fixed circle
(a) which cuts the x-axis orthogonally.
(b) with radius equal to b.
(c) on which the length of the tangent from the origin is $\sqrt{d^2 - b}$.
(d) with centre $(d, 0)$.

33. Point M moved along the circle $(x - 4)^2 + (y - 8)^2 = 20$, then it broke away from it and moving along a tangent to the circle cuts the x-axis at the point (–2, 0). The co-ordinate of the point on the circle at which the moving point broke away can be
(a) $\left(\frac{-3}{5}, \frac{46}{5}\right)$ (b) $\left(\frac{-2}{5}, \frac{44}{5}\right)$
(c) (6, 4) (d) (3, 5)

SECTION-III

Comprehension Type

Paragraph for Question No. 34 to 36

Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the center of the circumcircle of triangle PAB.

34. Range of $y = f(x)$ is
(a) [–2, 1] (b) [–1, 4]
(c) [0, 2] (d) [2, 3]

35. Fundamental period of $y = f(x)$ is
(a) 2π (b) 3π
(c) π (d) not defined

36. Which of the following is true?
(a) $f(x) = 4$ has real roots
(b) $f(x) = 1$ has real roots
(c) range of $y = f^{-1}(x)$ is $\left[-\frac{\pi}{4}+2, \frac{\pi}{4}+2\right]$
(d) none of these

Paragraph for Question No. 37 to 39

Given equation of two intersecting circles $S_1 = 0$ and $S_2 = 0$

Equation of family of circles passing through the intersection point's of $S_1 = 0$ and $S_2 = 0$ is

$S_1 + \lambda S_2 = 0$, (where $\lambda \neq -1$)

Equation of common chord is $S_1 - S_2 = 0$

Equation of chord of contact for circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with respect to external point (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

37. The equation of the circle described on the common chord of the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 + 2y = 0$ as diameter is
(a) $x^2 + y^2 + x - y = 0$ (b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 - x + y = 0$ (d) $x^2 + y^2 + x + y = 0$

38. Let P be any moving point on the circle $x^2 + y^2 - 2x - 1 = 0$. AB be the chord of contact of this point P with respect to the circle $x^2 + y^2 - 2x = 0$. The locus of the circumcentre of the triangle PAB (C being centre of the circles) is
(a) $2x^2 + 2y^2 - 4x + 1 = 0$
(b) $x^2 + y^2 - 4x + 2 = 0$
(c) $x^2 + y^2 - 4x + 1 = 0$
(d) $2x^2 + 2y^2 - 4x + 3 = 0$

39. The common chord of the circle $x^2 + y^2 + 6x + 8y - 7 = 0$ and a circle passing through the origin and touching the line $y = x$, always passes through the point
(a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (b) (1, 1)
(c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) none of these

Paragraph for Question No. 40 to 42

Given P, Q are two points on the curve $y = \log_{1/2}(x - 0.5) + \log_2 \sqrt{4x^2 - 4x + 1}$, P also lies on the circle $x^2 + y^2 = 10$. If Q lies inside the given circle such that its abscissa is integer then

40. $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ is equal to

(a) 4 (b) 8 (c) 2 (d) 10

41. Max $\{|\overrightarrow{PQ}|\}$ is equal to

(a) 1 (b) 2 (c) 4 (d) 5

42. Min $\{|\overrightarrow{PQ}|\} =$

(a) 1 (b) 3 (c) 4 (d) 6

Paragraph for Question No. 43 to 45

The line $x + 2y + a = 0$ intersects the circle $x^2 + y^2 - 4 = 0$ at two distinct points A and B. Another line $12x - 6y - 41 = 0$ intersect the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ at two distinct points C and D.

43. The value of a so that the line $x + 2y + a = 0$ intersect the circle $x^2 + y^2 - 4 = 0$ at two distinct points A and B is

(a) $-2\sqrt{5} < a < 2\sqrt{5}$ (b) $0 < a < 2\sqrt{5}$

(c) $-\sqrt{5} < a < \sqrt{5}$ (d) $0 < a < \sqrt{5}$

44. The value of 'a' for which the four points A, B, C and D are concyclic is

(a) 1 (b) 3 (c) 4 (d) 2

45. The equation of circle passing through the points A, B, C and D is

(a) $5x^2 + 5y^2 - 8x - 16y - 36 = 0$

(b) $5x^2 + 5y^2 + 8x - 16y - 36 = 0$

(c) $5x^2 + 5y^2 + 8x + 16y - 36 = 0$

(d) $5x^2 + 5y^2 - 8x - 16y + 36 = 0$

SECTION-IV

Matrix-Match Type

46. Match the following :

	Column I		Column II
A.	The number of common tangents that can be drawn to the two circles $C_1 : x^2 + y^2 - 4x - 6y - 3 = 0$ $C_2 : x^2 + y^2 + 2x + 2y + 1 = 0$ is	(p)	2
B.	The common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin an angle equal to π/k, then k = ?	(q)	0
C.	Shortest distance from the point $(2, -7)$ to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is	(r)	3
D.	If real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2 = (14)^2$, then the minimum value of $\sqrt{x^2 + y^2}$ is	(s)	1

47. Match the following :

	Column I		Column II
A.	If the circle $x^2 + y^2 + 2x + c = 0$ and $x^2 + y^2 + 2y + c = 0$ touch each other, then the set of 'c' values is contained in or equal to	(p)	the set satisfying $c = 1$
B.	If the circle $x^2 + y^2 + 2x + 3y + c = 0$ and $x^2 + y^2 - x + 2y + c = 0$ intersect orthogonally, then the set of 'c' values is contained in or equal to	(q)	the set satisfying $|c| < 2$
C.	If the circle $x^2 + y^2 = 9$ contains the circle $x^2 + y^2 - 2x + 1 - c^2 = 0$, then the set of '$c$' values is contained in or equal to	(r)	the set satisfying $c = 1/2$
D.	If the circle $x^2 + y^2 = 9$ is contained in the circle $x^2 + y^2 - 6x - 8y + 25 - c^2 = 0$, then the set of '$c$' values is contained in or equal to	(s)	the set satisfying $|c| > 8$
		(t)	the set satisfying $2 < |c| < 8$

48. Match the following:

	Column I		Column II
A.	The radical axis of two circles	(p)	subtends a right angle at a point of intersection
B.	The common tangent to two intersecting circles of equal radii	(q)	is perpendicular to the line joining the centres
C.	The common chord of two intersecting circles	(r)	is parallel to the line joining the centres
D.	The line joining the centres of two circles intersecting orthogonally	(s)	is bisected by the line joining the centres.

SECTION-V

Integer Answer Type

49. Let $S_1 \equiv x^2 + y^2 - 4x - 8y + 4 = 0$ and S_2 is its image in the line $y = x$. The radius of the circle touching $y = x$ at (1, 1) and orthogonal to S_2 is $\frac{3}{\sqrt{\lambda}}$, then $\lambda^2 + 2 =$

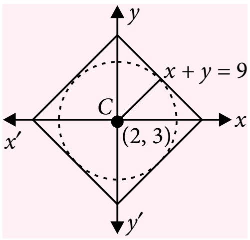
50. The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular then sum of all possible values of h/r is

51. The number of integral values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circle $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$ is

52. Let r be the radius of incircle of triangle formed by joining centres of $(x - a)^2 + (y - b)^2 = 9$, $(x - a)^2 + (y - b - 7)^2 = 16$ and circle touching above two circles and having radius 5 units. Find r^2.

SOLUTIONS

1. (d) : Perpendicular distance from centre to tangent = radius

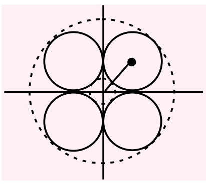


$$r = \frac{|2+3-9|}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Equation of circle is $(x - 2)^2 + (y - 3)^2 = 8$

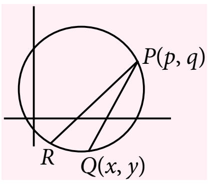
2. (a) : Radius of smallest circle is $r + a = a\sqrt{2}$



$\Rightarrow \quad r = a\sqrt{2} - a$

$\therefore$ The radius of another circle is $a\sqrt{2} + a$

3. (d) : Let PQ be a chord of the given circle passing through $P(p, q)$ and the coordinates of Q be (x, y). Since PQ is bisected by the x-axis, the mid-point of PQ lies on the x-axis which gives $y = -q$



Now Q lies on the circle $x^2 + y^2 - px - qy = 0$

So $x^2 + q^2 - px + q^2 = 0$

$\Rightarrow \quad x^2 - px + 2q^2 = 0 \qquad \ldots(i)$

Which gives two values of x and hence the coordinates of two points Q and R (say), so that the chords PQ and PR are bisected by x-axis. If the chords PQ and PR are distinct, the roots of (i) are real and distinct.

$\Rightarrow$ the discriminant $p^2 - 8q^2 > 0 \Rightarrow p^2 > 8q^2$.

4. (b) : Equation of any circle through (0, 0) and (1, 0) is

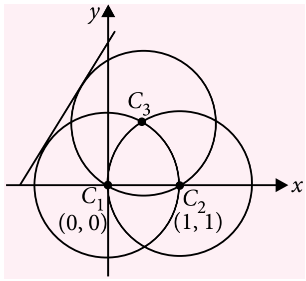
$$(x-0)(x-1)+(y-0)(y-0)+\lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}=0$$

$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$

If it represents C_3, its radius $=1$

$\Rightarrow \quad 1 = (1/4) + (\lambda^2/4)$

$\Rightarrow \quad \lambda = \pm\sqrt{3}$



As the centre of C_3, lies above the x-axis , we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$.

Since C_1 and C_3 intersect and are of unit radius. So their common tangents are parallel to the line joining their centres (0, 0) and $(1/2, \sqrt{3}/2)$.

So, let the equation of a common tangents be

$$\sqrt{3}x - y + k = 0.$$

It will touch C_1, if $\left|\frac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation be $\sqrt{3}x - y + 2 = 0$.

5. (c) : $C = \left(\frac{9}{2}, k\right), k > 0$

$$\left(x - \frac{9}{2}\right)^2 + (y-k)^2 = \frac{25}{4} + k^2$$

or $\quad x^2 + y^2 - 9x - 2ky + 14 = 0, k > 0$

6. (a) : $2x^2 + 6y^2 - x + y - 7xy - 1 = 0$

$x - 2y - 1 = 0 \qquad \ldots(1)$

and $\quad 2x - 3y + 1 = 0 \qquad \ldots(2)$

Centre is $(-5, -3)$

$\therefore$ The required equation of a circle be

$x^2 + y^2 + 10x + 6y - 19 = 0$.

7. (d) : Vertex $A = (0, -\sqrt{12})$

Centroid $G = \left(0, \frac{-2}{\sqrt{3}}\right)$, Circumradius $= \sqrt{4 + \frac{4}{3}} = \frac{4}{\sqrt{3}}$

$\therefore \quad \sqrt{3}(x^2 + y^2) + 4y - 4\sqrt{3} = 0$

8. (a) : $C = (6, 8)$ and radius $= 5 = AC$ (say)

$OC = \sqrt{36+64} = 10$, $OA = 5$

$\because \quad OA : AC = 5 : 5 = 1 : 1$

A is midpoint of OC *i.e.* (3, 4)

Let Coordinate of B be (h, k)

$\because$ 'C' is the midpoint of AB

$\therefore \quad h = 9, k = 12 \therefore B(9, 12)$

9. (a) : We have, $C = (a, a)$

Radius $= 4\sqrt{5}$ = length of the $\perp$ from (a, a) to the line

i.e., $\dfrac{|a+2(a)-3|}{\sqrt{4+1}} = \pm 4\sqrt{5} \Rightarrow a = \dfrac{23}{3}, \dfrac{-17}{3}$

$\therefore$ Centre $\left(\dfrac{23}{3}, \dfrac{23}{3}\right)$ or $\left(\dfrac{-17}{3}, \dfrac{-17}{3}\right)$

Only $C = \left(\dfrac{23}{3}, \dfrac{23}{3}\right)$ satisfies the given inequality

$3x + 6y > 10$

$\therefore \left(x - \dfrac{23}{3}\right)^2 + \left(y - \dfrac{23}{3}\right)^2 = 80$

10. (b) : $C\left(\dfrac{1}{2}, \dfrac{-3}{2}\right)$ and let $4 : y = mx$

$\because$ Perpendicular distance from centre to two equal chords are equal.

$\therefore$ Perpendicular distance from L_2 = Perpendicular distance from L_1

$\Rightarrow \dfrac{\left|\dfrac{1}{2} - \dfrac{3}{2} - 1\right|}{\sqrt{1+1}} = \dfrac{\left|\dfrac{m}{2} + \dfrac{3}{2}\right|}{\sqrt{1+m^2}} \Rightarrow m = 1, \dfrac{-1}{7}$

$\therefore$ Two chords are $y = x$ and $x + 7y = 0$.

11. (c) : $(x - p)(x - \alpha) + (y - q)(y - \beta) = 0$

or $x^2 + y^2 - (p + \alpha)x - (q + \beta)y + p\alpha + q\beta = 0$ ...(1)

Put $y = 0$, we get $x^2 - (p + \alpha)x + p\alpha + q\beta = 0$ ...(2)

$\therefore$ Locus of B (α, β) is $(p - x)^2 = 4qy$ or $(x - p)^2 = 4qy$

12. (c) : We have, $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

Chord of contact to the given circle is $xx_1 + yy_1 = a^2$

Since it passes through $Q(x_2, y_2)$

$\therefore$ $x_2x_1 + y_2y_1 = a^2$ ...(i)

Now, $l_1 = \sqrt{x_1^2 + y_1^2 - a^2}$, $l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$

and $PQ = \sqrt{l_1^2 + l_2^2}$

13. (b) : Let the three points be $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$

Chord of contacts to the given circle are :

$xx_1 + yy_1 = a^2$, $xx_2 + yy_2 = a^2$ and $xx_3 + yy_3 = a^2$

$\because$ These lines will be concurrent if

$$\begin{vmatrix} x_1 & y_1 & -a^2 \\ x_2 & y_2 & -a^2 \\ x_3 & y_3 & -a^2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_1 & -1 \\ x_2 & y_2 & -1 \\ x_3 & y_3 & -1 \end{vmatrix} = 0$$

which is the condition of the collinearity of A, B, C

14. (b) : Equation of circumcircle of a ΔABC is

$$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{17}{2}$$

$$C = \left(\frac{-3}{2} + \sqrt{\frac{17}{2}}\cos\theta, \frac{-1}{2} + \sqrt{\frac{17}{2}}\sin\theta\right)$$

Circumcentre of ΔABC is $\left(\dfrac{-3}{2}, \dfrac{-1}{2}\right)$.

Centroid can be obtained.

In a triangle centroid, circumcentre and orthocentre are collinear.

15. (b) : Let co-ordinates of $A(x_1, x_1)$ and $B(x_2, mx_2)$.

Clearly $(x_1 - x_2)^2 + (x_1 - mx_2)^2 = 16$

Let mid point of AB be $P(h, k)$

$\Rightarrow$ $x_1 + x_2 = 2h$ and $x_1 + mx_2 = 2k$

$\Rightarrow$ $(x_1 - x_2)^2 + 4x_1x_2 = 4h^2$ and

$(x_1 - mx_2)^2 + 4mx_1x_2 = 4k^2$

$(x_1 - x_2)^2 + (x_1 - mx_2)^2 = 4h^2 + 4k^2 = 16$,

when $m = -1$.

16. (c) : Both the circles have radius = 5 and they intersect each other, therefore their common tangent is parallel to the line joining their centres.

Equation of the line joining their centres is

$7x - 5y + 1 = 0$.

$\therefore$ Equation of the common tangent is $7x - 5y = c$

$\therefore \left|\dfrac{c+1}{\sqrt{74}}\right| = 5 \Rightarrow c = \pm 5\sqrt{74} - 1$

$\therefore$ Equation is $7x - 5y + 1 \pm 5\sqrt{74} = 0$.

17. (b) : $P(\cos\theta, 2\sin\theta)$, $Q(-2\cos\theta, -2\sin\theta)$

$$\alpha\beta = \frac{|2\cos\theta + 2\sin\theta||-2\cos\theta - 2\sin\theta - 1|}{2}$$

$$= \frac{|4(\cos\theta + \sin\theta)^2 - 1|}{2} \le \frac{7}{2}$$

18. (a) : $\tan\theta = \dfrac{r}{PA}$

Given $\dfrac{1}{r^2} + \dfrac{1}{(PA)^2} = \dfrac{1}{16} \Rightarrow \dfrac{\cot^2\theta + 1}{(PA)^2} = \dfrac{1}{16}$

$\Rightarrow$ $(PA)\sin\theta = 4$

$\Rightarrow$ Length of chord $= 2(PA)\sin\theta = 2 \times 4 = 8$.

19. (a) : Centres and radii of the given circles are $C_1 = (1, 3)$, $r_1 = r$ and $C_2 = (4, -1)$, $r_2 = 3$ respectively, since circles intersect in two distinct points, then

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow$ $|r - 3| < 5 < r + 3$ ...(i)

From last two relations, $r > 2$

From first two relations, $|r - 3| < 5$

$\Rightarrow$ $-5 < r - 3 < 5 \Rightarrow -2 < r < 8$ ...(ii)

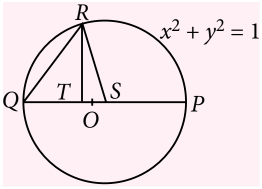
From equations (i) and (ii), we get $2 < r < 8$

20. (c) : Q is $(-1, 0)$

The circle with centre at Q and variable radius r has the equation $(x + 1)^2 + y^2 = r^2$

This circle meets the line segment QP at S where $QS = r$

It meets the circle $x^2 + y^2 = 1$ at $R\left(\frac{r^2-2}{2}, \frac{r}{2}\sqrt{4-r^2}\right)$ found by solving the equations of the two circles simultaneously.



A = area of the triangle QSR

$$=\frac{1}{2}QS\times RT$$

$$=\frac{1}{2}r\left(\frac{r}{2}\sqrt{4-r^2}\right) \text{ since } RT \text{ is the } y \text{ coordinate of } R$$

$$\frac{dA}{dr}=\frac{1}{4}\left\{2r\sqrt{4-r^2}+\frac{r^2(-r)}{\sqrt{4-r^2}}\right\}$$

$$=\frac{\{2r(4-r^2)-r^3\}}{4\sqrt{4-r^2}}=\frac{8r-3r^3}{4\sqrt{4-r^2}}$$

$\frac{dA}{dr}=0$, when $r(8 - 3r^2) = 0$, giving $r=\sqrt{\frac{8}{3}}$

$$\frac{d^2A}{dr^2}=\frac{4\sqrt{4-r^2}(8-9r^2)-(8r-3r^2)\frac{(-4r)}{\sqrt{4-r^2}}}{16(4-r^2)}$$

where, $r=\sqrt{\frac{8}{3}}$, $\frac{d^2A}{dr^2}<0$.

Hence A is maximum when $r=\sqrt{\frac{8}{3}}$ and the maximum area

$$=\frac{8}{4\times 3}\sqrt{4-\frac{8}{3}}=\frac{16}{12\sqrt{3}}=\frac{4}{3\sqrt{3}}=\frac{4\sqrt{3}}{9}$$

21. (d): Let (x_1, y_1) be the centre. Since it touches y-axis its radius is $|x_1|$. Also it touches the given circle externally.

$\therefore \quad \sqrt{(x_1-3)^2+(y_1-3)^2} = |x_1+2|$

Squaring both sides, we get

$$x_1^2 + y_1^2 - 6x_1 - 6y_1 + 18 = x_1^2 + 4x_1 + 4$$

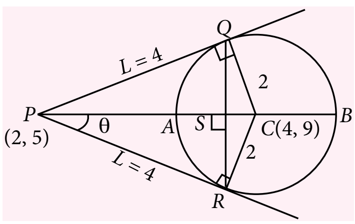
$\Rightarrow \quad y_1^2 - 10x_1 - 6y_1 + 14 = 0$

Thus, the locus of the centre of a circle at (x_1, y_1) is $y^2 - 10x - 6y + 14 = 0$

22. (a, b, c): We have, radius $=\sqrt{16+81-93}=2$

$$CP=\sqrt{20},\ AP=\sqrt{20}-2;\ BP=\sqrt{20}+2$$

$\Rightarrow \quad CP=\frac{AP+BP}{2} \Rightarrow$ (a) is correct



Now, let $L=PR=\sqrt{(PC)^2-r^2}=\sqrt{20-4}=4=PQ$

and $\tan\theta=\frac{2}{4}=\frac{1}{2}$

Also, $\cos\theta=\frac{PS}{PR}$

$\Rightarrow \quad PS=PR\cos\theta=4\cdot\left(\frac{2}{\sqrt{5}}\right)=\frac{8}{\sqrt{5}}$

Harmonic mean between PA and PB

$$=\frac{2(\sqrt{20}-2)(\sqrt{20}+2)}{2\sqrt{20}}=\frac{16}{2\sqrt{5}}=\frac{8}{\sqrt{5}}=PS$$

$\Rightarrow \quad$ (c) is correct

Hence $(PS)(PC)=\left(\frac{8}{\sqrt{5}}\right)(\sqrt{20})=16=(PR)^2$

$\Rightarrow \quad PR$ is the Geometric Mean of PS and PC

$\Rightarrow \quad$ (b) is correct

Now angle between the two tangents is 2θ, then

$$2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{2\cdot\frac{1}{2}}{1-\frac{1}{4}}\right) \quad \left(\text{As } \tan\theta=\frac{1}{2}\right)$$

$$=\tan^{-1}\left(\frac{4}{3}\right).$$

$\Rightarrow \quad$ (d) is incorrect.

23. (a, d): Let $Q = (3\cos\theta, 3\sin\theta)$, $N = (3\cos\theta, 0)$

Points of trisection are $(3\cos\theta, \sin\theta)$ and $(3\cos\theta, 2\sin\theta)$

Locus is $\frac{x^2}{9}+y^2=1;\ \frac{x^2}{9}+\frac{y^2}{4}=1$

24. (a, c): We have, $\frac{x_1-1}{\cos\theta}=\frac{y_1-2\sqrt{2}}{\sin\theta}=2$

Also, $x_1^2 + y_1^2 = 9$

On solving the above equations, we get

$$\cos\theta=\frac{7}{9} \text{ or } -1$$

Then the required points are

$$(-1,\ 2\sqrt{2}) \text{ and } \left(\frac{23}{9},\ \frac{10\sqrt{2}}{9}\right)$$

25. (a, b, c, d): $(BE)_{max}$ = diameter of circle $C_2 = 2r_2$

$$(BE)_{min}=2\sqrt{r_2^2-r_1^2}$$

$(OA^2 + OB^2 + BE^2)_{min}$ is

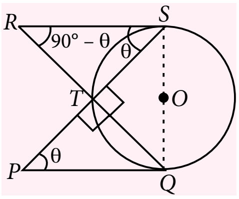
$$r_1^2+r_2^2+4r_2^2-4r_1^2=5r_2^2-3r_1^2$$

$(OA^2 + OB^2 + BE^2)_{max}$ is

$$r_1^2+r_2^2+4r_2^2=r_1^2+5r_2^2$$

26. (a, b, c) : $\underline{|QTS} = \underline{|PTQ} = \dfrac{\pi}{2}$ as QS is a diameter

$\because \quad \dfrac{QT}{PQ} = \sin\theta, \ \dfrac{ST}{RS} = \cos\theta,$

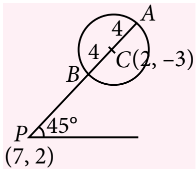


$\dfrac{4}{PQ} = \tan\theta$ and $\dfrac{4}{RS} = \cot\theta$

$\therefore \quad \left(\dfrac{QT}{PQ}\right)^2 + \left(\dfrac{ST}{RS}\right)^2 + PQ \cdot RS$

$= \sin^2\theta + \cos^2\theta + (4\cot\theta)(4\tan\theta)$

27. (a, c) : Slope of $PC = \dfrac{-3-2}{2-7} = 1$



$\because \quad \theta = 45° \ \therefore \ \tan\theta = 1$

Equation of PA is $\dfrac{x-7}{1/\sqrt{2}} = \dfrac{y-2}{1/\sqrt{2}} = r$

$\therefore \quad \left(7 + \dfrac{r}{\sqrt{2}}, 2 + \dfrac{r}{\sqrt{2}}\right)$ lie on circle, then,

$$\left(7+\frac{r}{\sqrt{2}}\right)^2 + \left(2+\frac{r}{\sqrt{2}}\right)^2 - 4\left(7+\frac{r}{\sqrt{2}}\right) + 6\left(2+\frac{r}{\sqrt{2}}\right) - 3 = 0$$

$\Rightarrow \quad r^2 + 10\sqrt{2}r + 34 = 0 \ \therefore \ r = -5\sqrt{2} \pm 4$

$\therefore$ Point $\left(7 + \dfrac{-5\sqrt{2} \pm 4}{\sqrt{2}}, 2 + \dfrac{-5\sqrt{2} \pm 4}{\sqrt{2}}\right)$

$\Rightarrow \quad (2 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2})$

taking –ve sign for A and +ve sign for B.

28. (a, c) : Radical axis of the given circles is $x + 5y - 6 = 0$, which passes through (1, 1)

Let the given circles intersect the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ orthogonally then

$2g(-3) + 2f(-6) = c + 1$ and $2g(-2) + 2f(-1) = c - 11$

$\Rightarrow \quad 2g(-1) + 2f(-5) = 12 \ \Rightarrow \ -g - 5f - 6 = 0$

$\Rightarrow$ The radical axis passes through the centre $(-g, -f)$ of the third circle.

Also, it does not pass through the mid-point of the line joining the centres.

29. (a, c) : $\because$ C_2 is the director circle of C_1.

$\therefore \quad$ Equation of C_2 is

$x^2 + y^2 = 2(2)^2 = 8$

Again C_3 is the director circle of C_2.

Hence the equation of C_3 is $x^2 + y^2 = 2(8) = 16$

30. (a, c, d) : $C_1 \equiv x^2 + y^2 - 2x - 4y - 4 = 0 \quad \ldots(1)$

and $C_2 = x^2 + y^2 + 2x + 4y + 4 = 0 \quad \ldots(2)$

$\therefore \quad$ Radical axis is $C_1 - C_2 = 0$

$\Rightarrow \quad -4x - 8y - 8 = 0$ or $x + 2y + 2 = 0$, which is $L = 0$.

$\therefore \quad$ (a) option is correct.

Now, centre and radius of $C_1 = 0$ are (1, 2) and 3 respectively.

$\therefore \quad$ length of $\perp$ from (1, 2) on $L = 0$ is

$\dfrac{|1+4+2|}{\sqrt{1+4}} = \dfrac{7}{\sqrt{5}} \neq$ radius

$\therefore \quad$ (b) option is wrong.

Now, L is also the common chord of C_1 and C_2

$\therefore \quad$ (c) option is correct.

$\because \quad$ Centres of $C_1 = 0$ and $C_2 = 0$ are (1, 2) and (–1, –2) respectively.

$\therefore \quad$ Slope of line joining centres of circles $C_1 = 0$ and $C_2 = 0$ is $\dfrac{-2-2}{-1-1} = \dfrac{4}{2} = 2 = m_1$ (say)

And slope of $L = 0$ is $-\dfrac{1}{2} = m_2$ (say)

Hence, L is perpendicular to the line joining centres of C_1 and C_2.

$\therefore \quad$ (d) option is correct.

31. (a, b, c) : $(x - 1)^2 + (y - 1)^2 + \lambda(x + y - 2) = 0$

$\Rightarrow \quad x^2 + y^2 + (\lambda - 2)x + (\lambda - 2)y + 2 - 2\lambda = 0 \quad \ldots(1)$

$x^2 + y^2 + 4x + 5y - 6 = 0 \quad \ldots(2)$

Eq. of common chord PQ is $S - S' = 0$

$\Rightarrow \quad (\lambda - 6)x + (\lambda - 7)y + 8 - 2\lambda = 0 \quad \ldots(3)$

Let $PQ||$ to the given line $x + y - 2 = 0$

$\Rightarrow \quad \dfrac{6-\lambda}{\lambda-7} = -1 \ \Rightarrow \ 6 = 7,$ which is impossible.

Let $PQ \perp$ to the given line $x + y - 2 = 0$

$\Rightarrow \quad \dfrac{6-\lambda}{\lambda-7} = 1 \ \Rightarrow \ \lambda = \dfrac{13}{2},$ which is possible.

But when $\lambda = \dfrac{13}{2}$, we can see that the circles (1) and (2) are not intersecting each other and their radical axis is perpendicular to the given line $x + y - 2 = 0$.

Now eq. (3) can be written as

$-6x - 7y + 8 + \lambda(x + y - 2) = 0,$

which is in the form $L_1 + \lambda L_2 = 0$.

Solving L_1 and L_2, we get (6, –4).

$\because \quad$ (c) option is true but not (d).

32. (a, c, d) : $al^2 + 2dl + 1 = bm^2$

$bl^2 + al^2 + 2dl + 1 = bm^2 + bl^2$

$d^2l^2 + 2dl + 1 = b(l^2 + m^2)$

$\Rightarrow \quad \left|\dfrac{dl+1}{\sqrt{l^2+m^2}}\right| = \sqrt{b}$

$\Rightarrow \quad$ A fixed circle with centre = $(d, 0)$ and radius = $\sqrt{b}$.

$\therefore \quad (x - d)^2 + y^2 = b$ is the required equation of a circle.

33. (b, c) : $x^2 + y^2 - 8x - 16y + 60 = 0$ ...(1)

Equation of chord of contact from $(-2, 0)$ is

$3x + 4y - 34 = 0$ ...(2)

Intersection of (1) and (2) is

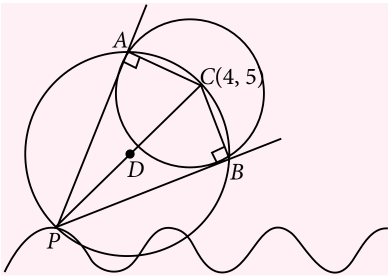
$$x^2 + \left(\frac{34-3x}{4}\right)^2 - 8x - 16\left(\frac{34-3x}{4}\right) + 60 = 0$$

$\Rightarrow$ $5x^2 - 28x - 12 = 0$ $\Rightarrow x = 6, -2/5$

$\therefore$ $(6, 4)$ and $\left(\frac{-2}{5}, \frac{44}{5}\right)$

34 - 36 :

34. (d) **35. (c)** **36. (c)**



Centre of the given circle is $C(4, 5)$. Points P, A, C, B are concyclic such that PC is a diameter of the circle. Hence, centre D of the circumcircle of ΔPAB is midpoint of PC, then we have

$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t + 5}{2}$$

Eliminating t, we have

$$k = \frac{\sin(2h-4)+5}{2} \text{ or } y = \frac{\sin(2x-4)+5}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{\sin^{-1}(2x-5)+4}{2}$$

Thus range of $y = \frac{\sin(2x-4)+5}{2}$ is $[2, 3]$ and period is π.

Also $f(x) = 4 \Rightarrow \sin(2x - 4) = 3$, which has no real solutions.

Also $f(x) = 1 \Rightarrow \sin(2x - 4) = -3$, which has no real solutions.

But range of $y = \frac{\sin^{-1}(2x-5)+4}{2}$ is $\left[-\frac{\pi}{4}+2, \frac{\pi}{4}+2\right]$

37 - 39 :

37. (d) : Equation of common chord is $2x - 2y = 0$

Equation of family of circle is

$x^2 + y^2 + 2x + \lambda(2x - 2y) = 0$

Centre of circle is $(-\lambda - 1, +\lambda)$

Centre lies on $y = x$

$\lambda = -\lambda - 1 \Rightarrow 2\lambda = -1 \Rightarrow \lambda = -1/2$

Equation is $x^2 + y^2 + x + y = 0$

38. (a)

39. (c) : Let the second circle

$x^2 + y^2 + 2gx + 2fy = 0$

Hence, $x^2 + y^2 + 2gx + 2fy = 0$ lies equal roots $f + g = 0$

Equation of common chord is

$2(g - 3)x + 2(-g - 4)y + 7 = 0$

$(-6x - 8y + 7) + g(2x - 2y) = 0$

Passes through the intersection point of

$-6x - 8y + 7 = 0$ and $2x - 2y = 0$

$$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \cos 60° = \frac{\sqrt{(h+1)^2 + (k-1)^2}}{2}$$

40 - 42 :

40. (a) **41. (b)** **42. (a)**

$$y = \log_{1/2}(x-0.5) + \log_2\sqrt{4x^2 - 4x + 1}$$

$= -\log(2x - 1) + \log_2 2 + \log(2x - 1)$

$\Rightarrow$ $y = 1$, $P(x, y)$ lies on the circle $x^2 + y^2 = 10$

$\Rightarrow$ $x = \pm 3$, neglecting $x = -3$

$\Rightarrow$ $Q(1, 1)$, $Q(2 , 1)$ lies inside the circle such that their abscissa is an integer .

43 - 45 :

43. (a) : If lines $x + 2y + a = 0$ intersect the circle $x^2 + y^2 = 4$, then

$$\left|\frac{0+0+a}{\sqrt{1+4}}\right| < 2 \Rightarrow -2\sqrt{5} < a < 2\sqrt{5}$$

44. (d)

45. (a) : Equation of the circle passing through the point of intersection of circle $x^2 + y^2 - 4 = 0$ and $x + 2y + 2 = 0$ is $x^2 + y^2 - 4 + \lambda(x + 2y + 2) = 0$ ...(1)

Common chord of circle represented by equation (1) and circle, $x^2 + y^2 - 4x - 2y + 1 = 0$ is

$(\lambda + 4)x + 2(\lambda + 1)y + 2\lambda - 5 = 0$ ...(2)

Since, equation (2) and $12x - 6y - 41 = 0$ represents the same line, we get $5x^2 + 5y^2 - 8x - 16y - 36 = 0$

46. A – r; B – p; C – p; D – s

(A) $C_1 = (2, 3)$ and $r_1 = \sqrt{4+9+3} = 4$

$C_2 = (-1, -1)$ and $r_2 = \sqrt{1+1-1} = 1$

$\Rightarrow$ $C_1C_2 = 5$ and $r_1 + r_2 = 5$

Also, $C_1C_2 = r_1 + r_2$

$\therefore$ No. of common tangents is 3.

(B) Common chord is $S_1 - S_2$

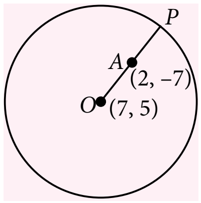
i.e., $x + y = 4$

$x + y = 4$ at origin subtend an angle of $\pi/2$.

Rvp

(C) $C = (7, 5)$

$r = \sqrt{49+25+151} = 15 = OP$



$OA = \sqrt{(7-2)^2 + (5-7)^2}$

$AP = OP - OA$

$AP = 15 - 13 = 2$

(D) Let $x + 5 = 14\cos\theta$

$y - 12 = 14\sin\theta$

$x^2 + y^2 = 365 + 28(12\sin\theta - 5\cos\theta)$

$(\sqrt{x^2+y^2})_{\min} = \sqrt{365 - 28\times 13} = 1$

47. A – q, r; B – p, q; C – q; D – s

(A) $c = 1/2$

(B) $c = 1$

(C) $C_1 = (0, 0)$, $C_2 = (1, 0)$

$r_1 = 3$, $r_2 = |c|$

$C_1C_2 < r_1 - r_2 \Rightarrow |c| < 2$

(D) $C_1 = (0, 0)$, $r_1 = 3$

$C_2 = (3, 4)$, $r_2 = |c| \Rightarrow |c| > 8$

48. A - q; B - r; C - q, s; D - p

Let the equations of the circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

(a) Equation of the radical axis is

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Slope of the radical axis $= -\dfrac{g_1 - g_2}{f_1 - f_2}$

Slope of the line joining the centres $= \dfrac{f_1 - f_2}{g_1 - g_2}$

So the radical axis is perpendicular to the line joining the centre.

(b) Common tangent to the intersecting circles of equal radii is at the same distance from the centres of the two circles and hence is parallel to the line joining the centres.

(c) Since the line joining the centres of the circles to the mid-point of the common chord is perpendicular to the chord as well as it bisects the chord.

(d) The line joining the centre of one to a point of intersection is tangent to the other circle.

So, by definition of orthogonality, they are perpendicular.

49. (6) : Centre of circle $S_1 = (2, 4)$

Centre of circle $S_2 = (4, 2)$

Radius of circle S_1 = radius of circle $S_2 = 4$

$\therefore$ Equation of circle S_2 is

$(x - 4)^2 + (y - 2)^2 = 16$

$\Rightarrow x^2 + y^2 - 8x - 4y + 4 = 0$ ...(1)

Equation of circle touching $y = x$ at (1, 1) can be taken as

$(x - 1)^2 + (y - 1)^2 + \lambda(x - y) = 0$

or $x^2 + y^2 + x(\lambda - 2) + y(-\lambda - 2) + 2 = 0$ ...(2)

As eqn. (2) is orthogonal to S_2

$\therefore 2\left(\dfrac{\lambda-2}{2}\right)\cdot(-4) + 2\left(\dfrac{-\lambda-2}{2}\right)\cdot(-2) = 4 + 2$

$\Rightarrow -4\lambda + 8 + 2\lambda + 4 = 6 \Rightarrow -2\lambda = -6 \Rightarrow \lambda = 3$

$\therefore$ Required equation of circle is

$x^2 + y^2 + x - 5y + 2 = 0.$

Radius $= \sqrt{\dfrac{1}{4} + \dfrac{25}{4} - 2} = \sqrt{\dfrac{26-8}{4}} = \sqrt{\dfrac{18}{4}} = \dfrac{3}{2}\sqrt{2}$

50. (0) : Combined equation of the tangents drawn from (0, 0) to the circle is

$(x^2 + y^2 - 2rx - 2hy + h^2)h^2 = (-rx - hy + h^2)^2$

Here coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow (h^2 - r^2) = 0 \Rightarrow \dfrac{h}{r} = \pm 1$

51. (1) : $S(x, y) = x^2 + y^2 - x - y - 6 = 0$ ...(1)

has centre at $C \equiv \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle.

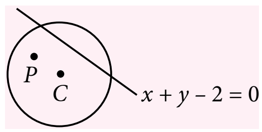
i.e. $S(\alpha - 1, \alpha + 1) < 0$

$\Rightarrow (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$

$\Rightarrow \alpha^2 - \alpha - 2 < 0$, *i.e.*, $(\alpha - 2)(\alpha + 1) < 0$

$\Rightarrow -1 < \alpha < 2$ ...(2)

Also P and C must lie on the same side of the line (see figure)



$L(x, y) \equiv x + y - 2 = 0$ ...(3)

i.e., $L(1/2, 1/2)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

Since $L\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{1}{2} + \dfrac{1}{2} - 2 < 0$

$\Rightarrow L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0,$

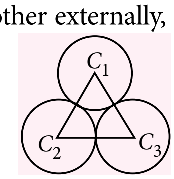
i.e., $\alpha < 1$ ...(4)

Inequalities (2) and (4) together give the permissible values of α as $-1 < \alpha < 1$.

52. (5) : $\because$ All three circles touch each other externally,

$\therefore C_1C_2 = 7,$

$C_2C_3 = 9$ and $C_3C_1 = 8$



Also, $s = \dfrac{7+8+9}{2} = 12$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{12\times5\times3\times4}$

$r = \dfrac{\Delta}{s} = \sqrt{5}$

CONCEPT BOOSTERS

Class XII

TRIGONOMETRY

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

ELEMENTS OF A TRIANGLE

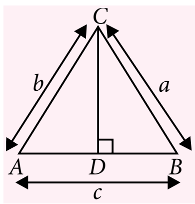
In a triangle ABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite to these angles are denoted by small letters a, b and c. Semi perimeter of the triangle is given by $s=\dfrac{a+b+c}{2}$ and its area is denoted by Δ.

- $A + B + C = 180°$
- $\Delta = \dfrac{1}{2}(\text{base})(\text{height})$

SINE RULE AND AREA OF TRIANGLE

- In ΔBCD,

  $CD = a \sin B = b \sin A$

  $\Rightarrow \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

  Similarly $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

  Hence in ΔABC,

  $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ (Sine rule)

  (where R is circumradius)

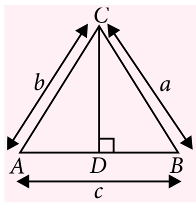
- Area of ΔABC,

  $\Delta = \dfrac{1}{2} \times AB \times CD = \dfrac{1}{2} \times c \times a \sin B = \dfrac{1}{2} ac \sin B$

  Similarly $\Delta = \dfrac{1}{2} bc \sin A = \dfrac{1}{2} ac \sin B = \dfrac{1}{2} ab \sin C.$

COSINE RULE

In ΔDBC



$BC^2 = DC^2 + DB^2$

$= DC^2 + (AB - AD)^2$

$= CD^2 + AD^2 + AB^2 - 2AB \cdot AD$

$= AC^2 + AB^2 - 2AB \cdot AC \cdot \cos A$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$

- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$
- $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

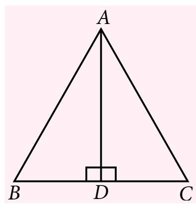
If $\angle A = 60°$, then $b^2 + c^2 - a^2 = bc$

If $\angle B = 60°$, then $a^2 + c^2 - b^2 = ac$

If $\angle C = 60°$, then $a^2 + b^2 - c^2 = ab$

PROJECTION FORMULAE

In ΔABC



$BC = BD + DC$

$= AB \cos B + AC \cos C$

$\Rightarrow \quad a = c \cos B + b \cos C$

- $a = b \cos C + c \cos B$
- $b = c \cos A + a \cos C$
- $c = a \cos B + b \cos A$

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NAPIER'S ANALOGY (TANGENT RULE)

From Sine formula,

$$\frac{b}{c}=\frac{\sin B}{\sin C}\Rightarrow\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\tan\frac{A}{2}.\tan\frac{B-C}{2}$$

$$\therefore\ \tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$$

- $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$
- $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$
- $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

HALF ANGLE FORMULAE

Since $2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a+b-c)(a-b+c)}{2bc}$$

$$=\frac{(2s-2c)(2s-2b)}{2bc}$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

- $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$
- $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$
- $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

Since $2\cos^2\frac{A}{2}=1+\cos A$

$$=1+\frac{b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}=\frac{2s(2s-2a)}{2bc}$$

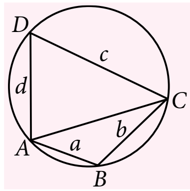
$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

- $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$
- $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$
- $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

Using $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$, we get

- $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$
- $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

AREA OF CYCLIC QUADRILATERAL



Let $ABCD$ be a cyclic quadrilateral whose sides AB, BC, CD and DA are respectively a, b, c and d.
Area of quadrilateral

$$=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Circumradius of a cyclic quadrilateral

$$R=\frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)}$$

CENTROID AND MEDIANS OF A TRIANGLE

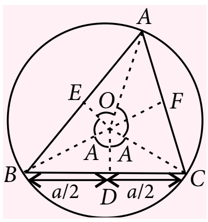
The line joining any vertex of a triangle to the mid point of the opposite side of the triangle is called median of the triangle. The three medians of a triangle are concurrent and the point of concurrency of the medians of any triangle is called the centroid of the triangle. The centroid divides the median in the ratio 2 : 1.

The distance of centroid from vertex A of triangle ABC, $GA=\frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and the distance of centroid from side BC of a triangle ABC, $G_a=\frac{2\Delta}{3a}$.

APOLLONIUS THEOREM :

In a triangle ABC, if AD is median through A, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$

CIRCUMCIRCLE



The circle which passes through the angular points of a ΔABC, is called its circumcircle. The centre of this circle *i.e.*, the point of concurrency of the

perpendicular bisectors of the sides of the ΔABC, is called the circumcentre.

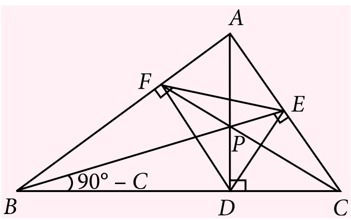
Radius of the circumcircle is given by the following formulae

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

Distance of circumcentre from vertex A of triangle $ABC = R$ and distance of circumcentre from side BC of triangle ABC is $R \cos A$.

ORTHOCENTRE AND PEDAL TRIANGLE OF A TRIANGLE.

In a triangle, the altitudes drawn from the three vertices to the opposite sides are concurrent and the point of cuncurrency of the altitudes of the triangle is called the orthocentre of the triangle.



$PD = 2R \cos B \cos C$ $\quad$ $PE = 2R \cos A \cos C$

$PF = 2R \cos A \cos B$ $\quad$ $PA = 2R \cos A$

$PB = 2R \cos B$ $\quad$ $PC = 2R \cos C$

$$AD = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A} = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$BE = \frac{1}{2}\sqrt{c^2 + a^2 + 2ac\cos B} = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$CF = \frac{1}{2}\sqrt{a^2 + b^2 + 2ab\cos C} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

The triangle formed by joining the feet of these perpendiculars is called the pedal triangle *i.e.*, ΔDEF is the pedal triangle of ΔABC.

Pedal triangle constructed on ΔABC w.r.t. orthocentre

- Its sides : $R \sin 2A$, $R \sin 2B$, $R \sin 2C$
- Its angles : $P - 2A$, $P - 2B$, $P - 2C$
- Its circumradius = $R/2$ and inradius = $2R \cos A \cos B \cos C$

BISECTORS OF THE ANGLES

If AD bisects the angle A and divide the base into portions x and y, we have, by Geometry,

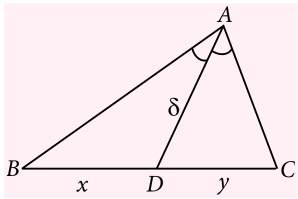
$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b} \quad \therefore \quad \frac{x}{c} = \frac{y}{b} = \frac{x+y}{b+c} = \frac{a}{b+c}$$

$$\Rightarrow \quad x = \frac{ac}{b+c} \text{ and } y = \frac{ab}{b+c}$$

Also, let δ be the length of AD

we have $\Delta ABD + \Delta ACD = \Delta ABC$

$$\therefore \quad \frac{1}{2}c\delta\sin\frac{A}{2} + \frac{1}{2}b\delta\sin\frac{A}{2} = \frac{1}{2}bc\sin A,$$

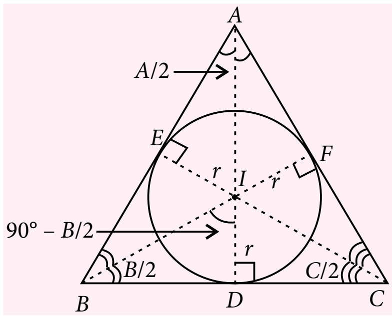


$$i.e., \quad \delta = \frac{bc}{b+c}\frac{\sin A}{\sin\frac{A}{2}}$$

$$= \frac{2bc}{b+c}\cos\frac{A}{2}$$

INCIRCLE

The circle which can be inscribed within the triangle so as to touch each of the sides of the triangle is called its incircle. The centre of this circle *i.e.*, the point of concurrency of angle bisectors of the triangle is called the incentre of the ΔABC.



Radius of the incircle is given by the following formulae :

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$r = \frac{a\sin(B/2)\sin(C/2)}{\cos A/2} = \frac{b\sin(A/2)\sin(C/2)}{\cos B/2}$$

$$= \frac{c\sin(B/2)\sin(A/2)}{\cos(C/2)}$$

INSCRIBED & CIRCUMSCRIBED POLYGONS (IMPORTANT FORMULAE)

- Area of Polygon of n sides inscribed in a circle of radius $R = \frac{1}{2}nR^2\sin\frac{2\pi}{n}$
- Area of Polygon of n sides inscribing a circle of radius $r = \frac{1}{2}nr^2\tan\frac{\pi}{n}$

- Side of Inscribed polygon $= 2R\sin\dfrac{\pi}{n}$
- Side of Circumscribed polygon $= 2r\tan\dfrac{\pi}{n}$

INVERSE TRIGONOMETRIC FUNCTION

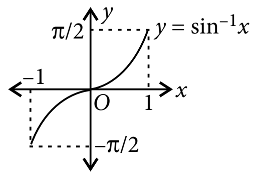
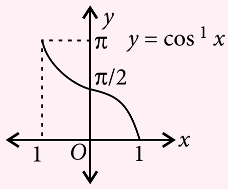
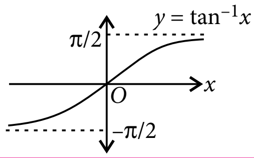
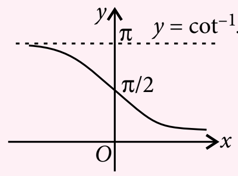
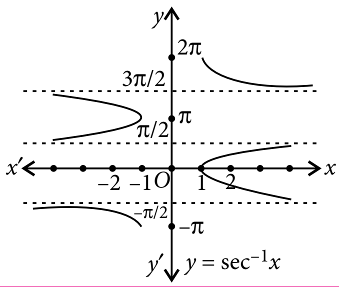
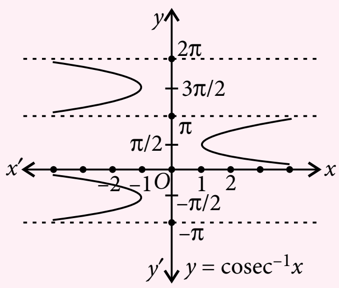
If $f : A \to B$ is one to one and onto function and g is a rule under which for every element $y \in B$, $\exists$ a unique element $x \in A$ then $g : B \to A$ is called Inverse function of $f : A \to B$.

i.e., $g = f^{-1}$

$\therefore \quad x = g(y) \quad \Rightarrow \quad x = f^{-1}(y)$

So $y = f(x)$ and $x = g(y)$ such that $f(g(y)) = y$ and $x = g(f(x))$ then f and g are said to be inverse function of each other.

DOMAIN, RANGE AND GRAPH OF INVERSE TRIGNOMETRIC FUNCTIONS

Function	Domain	Range	Graph
$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$	
$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$	
$y = \tan^{-1}x$	$x \in R$	$-\pi/2 < y < \pi/2$	
$y = \cot^{-1}x$	$x \in R$	$0 < y < \pi$	
$y = \sec^{-1}x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi,\ y \ne \pi/2$	
$y = \text{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$-\pi/2 \le y \le \pi/2,\ y \ne 0$	

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property I

- $\sin^{-1}(-x) = -\sin^{-1}x \ \forall\ x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1}x \ \forall\ x \in R$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x \ \forall\ x \in [-1, 1]$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x \ \forall\ x \in R$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

Property II

- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x \ \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}(1/x) = \begin{cases} \cot^{-1}x, x > 0 \\ -\pi + \cot^{-1}x, \ x < 0 \end{cases}$

Property III

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1, 1]$
- $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in R$

Property IV

- $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, xy < 1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy}, x > 0, \\ \qquad y > 0, xy > 1 \\ -\pi + \tan^{-1}\dfrac{x+y}{1-xy}, \ x < 0, \\ \qquad y < 0, xy > 1 \end{cases}$
- $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x-y}{1+xy}, xy > -1 \\ \pi + \tan^{-1}\dfrac{x-y}{1+xy}, x > 0, \\ \qquad y < 0 \ \& \ xy < -1 \\ -\pi + \tan^{-1}\dfrac{x-y}{1+xy}, \ x < 0, \\ \qquad y > 0 \ \& \ xy < -1 \end{cases}$

Remark : If $x_1, x_2, x_3, \ldots\ldots x_n \in R$, then

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots\ldots + \tan^{-1}x_n = \tan^{-1}\left(\frac{s_1 - s_3 + s_5 \ldots}{1 - s_2 + s_4 - s_6 \ldots}\right)$$

Where $s_1 = x_1 + x_2 + \ldots\ldots + x_n = \sum x_1$

$s_2 = x_1x_2 + x_2x_3 + \ldots\ldots + x_{n-1}x_n = \sum x_1x_2$

$s_3 = \sum x_1x_2x_3$ and so on.

Property V

- $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \\ \text{if } -1 \le x, y \le 1 \ \& \ x^2 + y^2 \le 1 \\ \text{or if } xy < 0 \ \& \ x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{\begin{array}{l} x\sqrt{1-y^2} \\ + y\sqrt{1-x^2} \end{array}\right\} \\ \text{if } 0 < x, y \le 1 \ \& \ x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{\begin{array}{l} x\sqrt{1-y^2} \\ + y\sqrt{1-x^2} \end{array}\right\} \\ \text{if } -1 \le x, y < 0 \ \& \ x^2 + y^2 > 1 \end{cases}$
- $\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \\ \text{if } -1 \le x, y \le 1 \ \& \ x^2 + y^2 \le 1 \\ \text{or if } xy > 0 \ \& \ x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{\begin{array}{l} x\sqrt{1-y^2} \\ - y\sqrt{1-x^2} \end{array}\right\} \\ \text{if } 0 < x \le 1, -1 \le y < 0 \\ \qquad \& \ x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{\begin{array}{l} x\sqrt{1-y^2} \\ - y\sqrt{1-x^2} \end{array}\right\} \\ \text{if } -1 \le x < 0, 0 < y \le 1 \\ \qquad \& \ x^2 + y^2 > 1 \end{cases}$

Property VI

- $$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } -1\le x,y\le 1\ \&\ x+y\ge 0\\ 2\pi-\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } -1\le x,y\le 1\ \&\ x+y\le 0\end{cases}$$

- $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } -1\le x,y\le 1, x\le y\\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)\\ \text{if } 0< x\le 1, -1\le y\le 0\ \&\ x\ge y\end{cases}$$

Property VII

- $$\sin^{-1}(2x\sqrt{1-x^2})\begin{cases}2\sin^{-1}x & \text{if } \frac{-1}{\sqrt2}\le x\le\frac{1}{\sqrt2}\\ \pi-2\sin^{-1}x, & \text{if } \frac{1}{\sqrt2}\le x\le 1\\ -\pi-2\sin^{-1}x & \text{if } -1\le x\le\frac{-1}{\sqrt2}\end{cases}$$

- $$\cos^{-1}(2x^2-1)=\begin{cases}2\cos^{-1}x \text{ if } 0\le x\le 1\\ 2\pi-2\cos^{-1}x \text{ if } -1\le x\le 0\end{cases}$$

- $$\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}2\tan^{-1}x \text{ if } -1\le x\le 1\\ \pi-2\tan^{-1}x \text{ if } x>1\\ -\pi-2\tan^{-1}x \text{ if } x<-1\end{cases}$$

- $$\cos^{-1}\frac{1-x^2}{1+x^2}=\begin{cases}2\tan^{-1}x \text{ if } x\ge 0\\ -2\tan^{-1}x \text{ if } x\le 0\end{cases}$$

- $$\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}2\tan^{-1}x, \text{ if } -1\le x\le 1\\ \pi-2\tan^{-1}x, \text{if } x<-1\\ -\pi+2\tan^{-1}x \text{ if } x>1\end{cases}$$

Property VIII

- $$\sin^{-1}(3x-4x^3)=\begin{cases}3\sin^{-1}x\ ; \frac{-1}{2}\le x\le\frac12\\ \pi-3\sin^{-1}x;\ \frac12\le x\le 1\\ -\pi-3\sin^{-1}x;\ -1\le x\le\frac{-1}{2}\end{cases}$$

- $$\cos^{-1}(4x^3-3x)=\begin{cases}-2\pi+3\cos^{-1}x;\ -1\le x\le\frac{-1}{2}\\ 2\pi-3\cos^{-1}x;\ \frac{-1}{2}\le x\le\frac12\\ 3\cos^{-1}x\ ;\ \frac12\le x\le 1\end{cases}$$

- $$\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\begin{cases}\pi+3\tan^{-1}x,\ x<\frac{-1}{\sqrt3}\\ 3\tan^{-1}x,\ \frac{-1}{\sqrt3}<x<\frac{1}{\sqrt3}\\ -\pi+3\tan^{-1}x,\ x>\frac{1}{\sqrt3}\end{cases}$$

Property IX : Conversion

- $$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$
$$=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\frac1x\right)$$

- $$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=$$
$$\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\frac1x\right)=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

- $$\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$
$$=\cot^{-1}\left(\frac1x\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

SECTION-I

Single Correct Answer Type

1. If $x=\log\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$, then $\sinh x=$

(a) $\tan 2\theta$ (b) $\cot 2\theta$
(c) $-\tan 2\theta$ (d) $-\cot 2\theta$

2. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is $60°$. He moves away from the pole along the line BC to a point D such that CD = 7 m. From D the angle of

elevation of the point A is 45°. Then the height of the pole is

(a) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)\text{m}$ (b) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)\text{m}$

(c) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}\text{m}$ (d) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}\text{m}$

3. The angle of elevation of a cloud from a point h m above the surface of a lake is θ and the angle of depression of its reflection in the lake is ϕ. The height of the cloud is

(a) $\frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$ (b) $\frac{h\sin(\phi-\theta)}{\sin(\phi+\theta)}$

(c) $\frac{h\sin(\theta+\phi)}{\sin(\theta-\phi)}$ (d) $\frac{h\sin(\theta-\phi)}{\sin(\theta+\phi)}$

4. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then

$\sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$ equals

(a) $\cos y$ (b) $\sin y$ (c) $\sin 2y$ (d) 0

5. If x_1, x_2, x_3, ..., x_n are in A.P, whose common difference is α, then the value of

$$\sin\alpha\,[\sec x_1\sec x_2+\sec x_2\sec x_3+.....+\sec x_{n-1}\sec x_n]=$$

(a) $\frac{\sin n\alpha}{\cos x_n\cos x_1}$ (b) $\frac{\sin(n-1)\alpha}{\cos x_n\cos x_1}$

(c) $\frac{\sin(n+1)\alpha}{\cos x_n\cos x_1}$ (d) $\frac{\cos(n-1)\alpha}{\cos x_n\cos x_1}$

6. If $a\sin^2x+b\cos^2x=c$, $b\sin^2y+a\cos^2y=d$ and $a\tan x=b\tan y$, then $\frac{a^2}{b^2}=$

(a) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$ (b) $\frac{(a+d)(c+a)}{(b+c)(d+b)}$

(c) $\frac{(a-d)(b-a)}{(a-c)(c-b)}$ (d) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$

7. If $\cos^3x\sin 2x=\sum_{r=0}^{n}a_r\sin(rx),\forall x\in R$ then

(a) $n=5, a_1=\frac{1}{2}$ (b) $n=5, a_1=\frac{1}{4}$

(c) $n=5, a_2=\frac{1}{8}$ (d) $n=5, a_2=\frac{1}{4}$

8. If $\cos\theta=\frac{a\cos\phi+b}{a+b\cos\phi}$, then $\tan\theta/2$ is equal to

(a) $\sqrt{\left(\frac{a-b}{a+b}\right)}\tan(\phi/2)$ (b) $\sqrt{\left(\frac{a+b}{a-b}\right)}\cos(\phi/2)$

(c) $\sqrt{\left(\frac{a-b}{a+b}\right)}\sin(\phi/2)$ (d) none of these

9. The value of $\sum_{r=0}^{10}\cos^3\frac{\pi r}{3}$ is equal to

(a) $\frac{-9}{2}$ (b) $\frac{-7}{2}$ (c) $\frac{-9}{8}$ (d) $\frac{-1}{8}$

10. The value of $\sin^3 10^0+\sin^3 50^0-\sin^3 70^0$ is equal to

(a) $-\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $-\frac{3}{4}$ (d) $-\frac{3}{8}$

11. If $\tan(\alpha-\beta)=\frac{\sin 2\beta}{3-\cos 2\beta}$, then

(a) $\tan\alpha=2\tan\beta$ (b) $\tan\beta=2\tan\alpha$

(c) $2\tan\alpha=3\tan\beta$ (d) $3\tan\alpha=2\tan\beta$

12. In a triangle ABC, if angle C is obtuse and angles A and B are given by roots of the equation $\tan^2x+p\tan x+q=0$, then the value of q is

(a) greater than 1 (b) less than 1

(c) equal to 1 (d) 0

13. If $2\sin x-\cos 2x=1$, then $\cos^2x+\cos^4x$ is equal to

(a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

14. If $ABCD$ is a cyclic quadrilateral such that $13\cos A+12=0$ and $3\tan B-4=0$, then the quadratic equation whose roots are $\tan C$ and $\cos D$ is

(a) $15x^2+60x-11=0$ (b) $60x^2+11x-15=0$

(c) $11x^2+60x-15=0$ (d) none of these

15. If A, B, C are the angles of a triangle such that $\cot\frac{A}{2}=3\tan\frac{C}{2}$, then $\sin A$, $\sin B$, $\sin C$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

16. If $\frac{2\tan\alpha}{1+\sec\alpha+\tan\alpha}=\lambda$, then $\frac{2\tan\alpha/2}{1+\tan\alpha/2}$ is equal to

(a) $\frac{1}{\lambda}$ (b) λ (c) $1-\lambda$ (d) $1+\lambda$

17. If $0\le A, B, C\le\pi$ and $A+B+C=\pi$, then the minimum value of $\sin 3A+\sin 3B+\sin 3C$ is

(a) -2 (b) $-\frac{3\sqrt{3}}{2}$

(c) 0 (d) none of these

18. The value of x which satisfies the equation

$2\tan^{-1}2x = \sin^{-1}\dfrac{4x}{1+4x^2}$ is

(a) $\left[\dfrac{1}{2},\infty\right)$ (b) $\left(-\infty,-\dfrac{1}{2}\right]$

(c) $[-1, 1]$ (d) $\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

19. Number of integral solutions of the equation

$3\tan^{-1}x + \cos^{-1}\left(\dfrac{1-3x^2}{(1+x^2)^{3/2}}\right) = 0$ is

(a) 1 (b) 2 (c) 0 (d) infinite

20. In a triangle ABC, with $A = \dfrac{\pi}{7}, B = \dfrac{2\pi}{7}, C = \dfrac{4\pi}{7}$, then $a^2 + b^2 + c^2$ is (R = circumradius of ΔABC)

(a) $4R^2$ (b) $6R^2$ (c) $7R^2$ (d) $8R^2$

21. For what value of x, $\sin(\cot^{-1}(x + 1)) = \cos(\tan^{-1} x)$?

(a) $\dfrac{1}{2}$ (b) 0 (c) 1 (d) $-\dfrac{1}{2}$

22. If the equation $x^2 + 12 + 3\sin(a + bx) + 6x = 0$ has atleast one real solution where $a, b \in [0, 2\pi]$, then value of $\cos\theta$, where θ is least positive value of $a + bx$ is

(a) π (b) 2π (c) 0 (d) $\dfrac{\pi}{2}$

23. In any ΔABC, which is not right angled, $\Sigma\cos A\ \text{cosec}B\ \text{cosec}C$ is

(a) constant (b) less than 1

(c) greater than 2 (d) none of these

24. If $\angle C = 90°$ in ΔABC, then

$\tan^{-1}\left(\dfrac{a}{b+c}\right) + \tan^{-1}\left(\dfrac{b}{c+a}\right)$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) π

SECTION-II

Multiple Correct Answer Type

25. In ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$ then $\angle R$ can be

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{3\pi}{4}$ (c) $\dfrac{5\pi}{6}$ (d) $\dfrac{\pi}{4}$

26. If $\left(\dfrac{\sin\theta}{\sin\phi}\right)^2 = \dfrac{\tan\theta}{\tan\phi} = 3$, then

(a) $\tan\phi = \dfrac{1}{\sqrt{3}}$ (b) $\tan\phi = -\dfrac{1}{\sqrt{3}}$

(c) $\tan\theta = \sqrt{3}$ (d) $\tan\theta = -\sqrt{3}$

27. Which of the following quantities are rational?

(a) $\sin\left(\dfrac{11\pi}{12}\right)\sin\left(\dfrac{5\pi}{12}\right)$

(b) $\text{cosec}\left(\dfrac{9\pi}{10}\right)\sec\left(\dfrac{4\pi}{5}\right)$

(c) $\sin^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{\pi}{8}\right)$

(d) $\left(1+\cos\dfrac{2\pi}{9}\right)\left(1+\cos\dfrac{4\pi}{9}\right)\left(1+\cos\dfrac{8\pi}{9}\right)$

28. A solution (x, y) of the system of equations $x - y = \dfrac{1}{3}$ and $\cos^2(\pi x) - \sin^2(\pi y) = \dfrac{1}{2}$ is

(a) $\left(\dfrac{7}{6},\dfrac{5}{6}\right)$ (b) $\left(\dfrac{2}{3},\dfrac{1}{3}\right)$

(c) $\left(\dfrac{-5}{6},\dfrac{-7}{6}\right)$ (d) $\left(\dfrac{13}{6},\dfrac{11}{6}\right)$

29. If $0 \le x \le 2\pi$ then $2^{\text{cosec}^2 x}\sqrt{\dfrac{y^2}{2} - y + 1} \le \sqrt{2}$ is

(a) satisfied by exactly one value of y

(b) satisfied by exactly two values of x

(c) satisfied by x for which $\cos x = 0$

(d) satisfied by x for which $\sin x = 0$

30. If $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2a$ and $2\sin\left(\dfrac{\alpha}{2}\right)\sin\left(\dfrac{\beta}{2}\right) = 1$ then

(a) $y^2 = 4a(a - x)$

(b) $\cos\alpha + \cos\beta = \cos\alpha\cos\beta$

(c) $\cos\alpha\cdot\cos\beta = \dfrac{4a^2 + y^2}{x^2 + y^2}$

(d) $\cos\alpha + \cos\beta = \dfrac{4ax}{x^2 + y^2}$

31. If $\dfrac{\tan 3A}{\tan A} = k\ (k \ne 1)$, then

(a) $\dfrac{\cos A}{\cos 3A} = \dfrac{k-1}{2}$ (b) $\dfrac{\sin 3A}{\sin A} = \dfrac{2k}{k-1}$

(c) $k < \dfrac{1}{3}$ (d) $k > 3$

32. Let $f_n(\theta)=\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)$ $(1+\sec 4\theta)....(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$

(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

33. Sum of series

$$\sum_{r=1}^{n}\sin^{-1}\left[\frac{2r+1}{r(r+1)\left(\sqrt{r^2+2r}+\sqrt{r^2-1}\right)}\right]$$ is

(a) $\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{n+1}\right)$ (b) $\cos^{-1}\left(\frac{1}{n+1}\right)$

(c) $\cos^{-1}\left(\frac{1}{n+2}\right)$ (d) none of these

34. Minimum positive values of x and y such that $x+y=\frac{\pi}{2}$ and $\sec x+\sec y=2\sqrt{2}$ are

(a) $x=\frac{\pi}{4}$ (b) $y=\frac{\pi}{4}$

(c) $x=-\frac{\pi}{4}$ (d) $y=-\frac{\pi}{4}$

SECTION-III

Comprehension Type

Paragraph for Question No. 35 to 37

Let $\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7}$ are the roots of equation $8x^3-4x^2-4x+1=0$

35. The value of $\sec\left(\frac{\pi}{7}\right)+\sec\left(\frac{3\pi}{7}\right)+\sec\left(\frac{5\pi}{7}\right)$ is

(a) 2 (b) 4
(c) 8 (d) none of these

36. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

37. The value of $\cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{2}$ (d) $\frac{\sqrt{7}}{8}$

SECTION-IV

Matrix-Match Type

38. Match the following :

	Column I		Column II
(A)	If $\tan\theta$ is the G.M. between $\sin\theta$ and $\cos\theta$, then $2-4\sin^2\theta+3\sin^4\theta-\sin^6\theta=$	(p)	1
(B)	$\sqrt{3}\cot 20^0-4\cos 20^0=$	(q)	0
(C)	$\cot 16^0\cot 44^0+\cot 44^0\cot 76^0-\cot 76^0\cot 16^0=$	(r)	3
(D)	$\sum_{r=1}^{9}\sin^2\left(\frac{r\pi}{18}\right)=$	(s)	5

39. Match the following trigonometric ratios with the equations whose one of the roots is given :

	Column I		Column II
(A)	$\cos 20^0$	(p)	$x^3-3x^2-3x+1=0$
(B)	$\sin 10^0$	(q)	$32x^5-40x^3+10x-1=0$
(C)	$\tan 15^0$	(r)	$8x^3-6x-1=0$
(D)	$\sin 6^0$	(s)	$8x^3-6x+1=0$

40. Match the following :

	Column I		Column II
(A)	If $\sin\theta=3\sin(\theta+2\alpha)$, then the value of $\tan(\theta+\alpha)+2\tan\alpha$ is	(p)	0
(B)	If $p\sin\theta+q\cos\theta=a$ and $p\cos\theta-q\sin\theta=b$, then $\frac{p+a}{q+b}+\frac{q-b}{p-a}+1$ is equal to	(q)	1
(C)	Then value of the expression $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is	(r)	$\sec\theta$
(D)	If $\sec\theta+\tan\theta=1$, then one root of the equation $(a-2b+c)x^2+(b-2c+a)x+(c-2a+b)=0$ is	(s)	$-\frac{1}{4}$
		(t)	$-\frac{1}{2}$

Rvp

SECTION-V

Integer Answer Type

41. Let $f(x) = 0$ be an equation of degree six, having integer coefficients and whose one root is $2\cos\frac{\pi}{18}$. Then, the sum of all the roots of $f'(x) = 0$, is

42. If $\cos\theta + \cos^2\theta + \cos^3\theta = 1$ and $\sin^6\theta = a + b\sin^2\theta + c\sin^4\theta$, then $a + b + c =$

43. If $\dfrac{\sin\theta}{\cos 3\theta} + \dfrac{\sin 3\theta}{\cos 9\theta} + \dfrac{\sin 9\theta}{\cos 27\theta} = \dfrac{1}{A}[\tan B\theta - \tan C\theta]$, then $(27A - B - 27C) =$

44. If $\dfrac{\tan x}{2} = \dfrac{\tan y}{3} = \dfrac{\tan z}{5}$ and $x + y + z = \pi$, $\tan^2 x + \tan^2 y + \tan^2 z = \dfrac{38}{K}$, then $K =$

45. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then the value of $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta$ must be

46. If $\alpha = \dfrac{\pi}{14}$, then the value of $(\tan\alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 4\alpha \tan\alpha)$ is

47. In a triangle ABC, if r_1, r_2, r_3 are the ex-radius and $\dfrac{bc}{r_1} + \dfrac{ac}{r_2} + \dfrac{ab}{r_3} = k\dfrac{abc}{2\Delta}\left[\dfrac{s}{a} + \dfrac{s}{b} + \dfrac{s}{c} - 3\right]$, then k is equal to

48. If $\alpha + \beta + \gamma = \pi$ and

$$\tan\left[\frac{\alpha+\beta-\gamma}{4}\right]\tan\left[\frac{\gamma+\alpha-\beta}{4}\right]\tan\left[\frac{\gamma+\beta-\alpha}{4}\right] = 1,$$

then the value of $1 + \cos\alpha + \cos\beta + \cos\gamma$ is $K - 1$, where K is

SOLUTIONS

1. (c) : $x = \log[\cot(\pi/4 + \theta)]$

$$= \log\left[\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right] \Rightarrow e^x = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\frac{(\cos\theta - \sin\theta)^2 - (\cos\theta + \sin\theta)^2}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}\right]$$

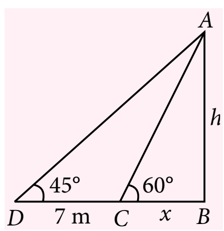
$$= \frac{1}{2}\left[\frac{-4\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}\right] = \frac{-\sin 2\theta}{\cos 2\theta} = -\tan 2\theta$$

2. (a) : $x = h\cot 60° = h/\sqrt{3}$

$x + 7 = h\cot 45°$

$\Rightarrow h - h/\sqrt{3} = 7$

$\Rightarrow h = \dfrac{7\sqrt{3}}{\sqrt{3} - 1}$ m

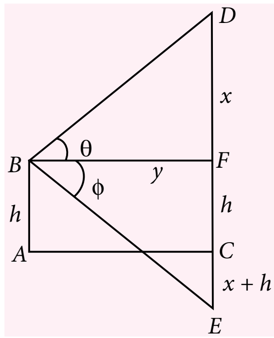


3. (a) : $\tan\theta = \dfrac{x}{y}$

$$\tan\phi = \frac{2h + x}{y}$$

$$\Rightarrow x = \frac{2h}{\cot\theta \cdot \tan\phi - 1}$$

$$CD = h + x = \frac{h\sin(\phi + \theta)}{\sin(\phi - \theta)}$$



4. (b) : $\dfrac{1 + \tan\frac{y}{2}}{1 - \tan\frac{y}{2}} = \left(\dfrac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right)^3$

Squaring both sides, we get $\dfrac{1 + \sin y}{1 - \sin y} = \dfrac{(1 + \sin x)^3}{(1 - \sin x)^3}$

Using componendo and dividendo

$$\frac{2\sin y}{2} = \frac{(3 + \sin^2 x)}{1 + 3\sin^2 x}\sin x$$

5. (b) : $\sin\alpha(\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + ... + \sec x_{n-1}\sec x_n)$

$$= \frac{\sin(x_2 - x_1)}{\cos x_1 \cos x_2} + \frac{\sin(x_3 - x_2)}{\cos x_2 \cos x_3} + ... + \frac{\sin(x_n - x_{n-1})}{\cos x_{n-1}\cos x_n}$$

$= \tan x_2 - \tan x_1 + \tan x_3 - \tan x_2 + ... + \tan x_n - \tan x_{n-1}$

$$= \tan x_n - \tan x_1 = \frac{\sin(x_n - x_1)}{\cos x_n \cos x_1} = \frac{\sin(n-1)\alpha}{\cos x_n \cos x_1}$$

6. (a) : $a\tan^2 x + b = c(1 + \tan^2 x)$

$$\Rightarrow \tan^2 x = \left(\frac{c - b}{a - c}\right), \ \tan^2 y = \left(\frac{d - a}{b - d}\right)$$

$$\frac{a^2}{b^2} = \frac{\tan^2 y}{\tan^2 x} = \frac{(a - d)(c - a)}{(b - c)(d - b)}$$

7. (b) : $\cos^3 x \sin 2x = \cos^2 x \cos x \sin 2x$

$$= \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{2\sin 2x \cos x}{2}\right)$$

$$= \frac{1}{4}(1 + \cos 2x)(\sin 3x + \sin x)$$

$$= \frac{1}{4}\left[\sin 3x + \sin x + \frac{1}{2}(2\sin 3x \cos 2x) + \frac{1}{2}(2\cos 2x \sin x)\right]$$

$$= \frac{1}{4}\left[\sin 3x + \sin x + \frac{1}{2}(\sin 5x + \sin x) + \frac{1}{2}(\sin 3x - \sin x)\right]$$

$$= \frac{1}{4}\left[\sin x + \frac{3}{2}\sin 3x + \frac{1}{2}\sin 5x\right]$$

By comparing, we get $n = 5$, $a_1 = \dfrac{1}{4}$

8. **(a) :** $\tan\theta/2=\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$

$$=\sqrt{\frac{1-\left(\frac{a\cos\phi+b}{a+b\cos\phi}\right)}{1+\left(\frac{a\cos\phi+b}{a+b\cos\phi}\right)}}=\sqrt{\frac{(a-b)(1-\cos\phi)}{(a+b)(1+\cos\phi)}}$$

$$=\sqrt{\frac{a-b}{a+b}}\tan(\phi/2)$$

9. **(d) :** Let $I=\sum_{r=0}^{10}\frac{1}{4}\left(\cos 3\frac{\pi r}{3}+3\cos\frac{\pi r}{3}\right)$

$$=\sum_{r=0}^{10}\frac{1}{4}\left(\cos\pi r+3\cos\frac{\pi r}{3}\right)=\frac{1}{4}(I_1+I_2)$$

$$\therefore\ I_1=\sum_{r=0}^{10}\cos\pi r=1-1+1-1+.....-1+1=1$$

$$I_2=3\sum_{r=0}^{10}\cos\frac{\pi r}{3}=3\left[1+\frac{\cos\frac{11}{2}\cdot\frac{\pi}{3}\sin\frac{10\pi}{6}}{\sin\frac{\pi}{6}}\right]$$

$$=3\left(1-\frac{3}{2}\right)=-\frac{3}{2}\Rightarrow I=\frac{1}{4}\left(1-\frac{3}{2}\right)=-\frac{1}{8}$$

10. **(d) :** We have, $\sin^3 10^0+\sin^3 50^0-\sin^3 70^0$

$$=\frac{1}{4}[(3\sin 10^0-\sin 30^0)+(3\sin 50^0-\sin 150^0)-(3\sin 70^0-\sin 210^0)]$$

$$=\frac{1}{4}\left[3(\sin 10^0+\sin 50^0-\sin 70^0)-\frac{3}{2}\right]$$

$$=\frac{1}{4}\left[3(\sin 10^0-2\cos 60^0\cdot\sin 10^0)-\frac{3}{2}\right]=-\frac{3}{8}$$

11. **(a) :** We have $\frac{\sin 2\beta}{3-\cos 2\beta}=\frac{2\sin\beta\cdot\cos\beta}{2-2\cos 2\beta+1+\cos 2\beta}$

$$=\frac{2\sin\beta\cdot\cos\beta}{4\sin^2\beta+2\cos^2\beta}=\frac{\tan\beta}{1+2\tan^2\beta}=\frac{2\tan\beta-\tan\beta}{1+2\tan^2\beta}\quad ...(i)$$

$$\because\ \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}\quad ...(ii)$$

$\therefore$ By (i) and (ii), we get

$$\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}=\frac{2\tan\beta-\tan\beta}{1+2\tan^2\beta}\Rightarrow \tan\alpha=2\tan\beta$$

12. **(b) :** We have $A+B=\pi-C$

$\Rightarrow\ \tan(A+B)=-\tan C$

$$\Rightarrow\frac{\tan A+\tan B}{1-\tan A\cdot\tan B}>0\ \left[\because \tan A>0,\tan B>0,\tan C<0\right]$$

$=\tan A\cdot\tan B<1$

$\Rightarrow\ q<1$

13. **(a) :** Given, $2\sin x+2\sin^2 x-1=1$

or $\sin^2 x+\sin x-1=0$

$$\therefore\ \sin x=\frac{-1\pm\sqrt{5}}{2}$$

Consider, $\sin x=\frac{-1+\sqrt{5}}{2}$

$$\Rightarrow\ \sin^2 x=\frac{3-\sqrt{5}}{2}\Rightarrow\cos^2 x=\frac{\sqrt{5}-1}{2}$$

$$\therefore\ \cos^2 x(1+\cos^2 x)=\frac{\sqrt{5}-1}{2}\times\frac{\sqrt{5}+1}{2}=1$$

14. **(b) :** In a cyclic quadrilateral, no angle is greater than 180°

Here $\cos A=-\frac{12}{13}\Rightarrow\frac{\pi}{2}<A<\pi$ and $0<C<\pi/2$

(since $A+C=180°$)

$$\therefore\ \tan A=-\frac{5}{12}\Rightarrow\tan C=\frac{5}{12}$$

Also $\tan B=\frac{4}{3}\Rightarrow 0<B<\frac{\pi}{2}$ and $\frac{\pi}{2}<D<\pi$

(since $B+D=180°$)

$$\therefore\ \cos B=\frac{3}{5}\Rightarrow\cos D=-\frac{3}{5}$$

Now, the required equation is

$$x^2-\left(\frac{5}{12}-\frac{3}{5}\right)x+\left(\frac{5}{12}\right)\left(-\frac{3}{5}\right)=0$$

$\Rightarrow\ 60x^2+11x-15=0$

15. **(a) :** Given $\cot\frac{A}{2}\cdot\cot\frac{C}{2}=3$

$$\Rightarrow\ \frac{\cos\frac{A}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}=3\Rightarrow\frac{\cos\frac{A-C}{2}}{\cos\frac{A+C}{2}}=2$$

(using componendo and dividendo)

$$\Rightarrow\ \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{2\sin\frac{A+C}{2}\cdot\cos\frac{A+C}{2}}=2\Rightarrow 2\sin B=\sin A+\sin C$$

16. (b) : We have $\dfrac{2\tan\alpha}{1+\sec\alpha+\tan\alpha}=\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$

$$=2\frac{2\tan\alpha/2}{(1+\tan^2\alpha/2)+(1-\tan^2\alpha/2)+2\tan\alpha/2}$$

$$=\frac{2\tan\alpha/2}{1+\tan\alpha/2}$$

17. (a) : Since $A+B+C=\pi$

$\Rightarrow$ All of $\sin 3A$, $\sin 3B$, $\sin 3C$ can't be negative.

Let us take $\sin 3A = -1 \Rightarrow A=\pi/2$

$\Rightarrow \sin 3A=-1$, $\sin 3B=-1$ and $\sin 3C = 0$

So minimum value is -2.

18. (d) : $-\dfrac{\pi}{2}\le 2\tan^{-1}2x\le\dfrac{\pi}{2} \Rightarrow -\dfrac{1}{2}\le x\le\dfrac{1}{2}$

19. (b) : Let $\tan^{-1}x=\theta, \theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$3\theta+\cos^{-1}(\cos 3\theta)=0$

$\cos^{-1}(\cos 3\theta)=-3\theta \Rightarrow -\pi\le 3\theta\le 0$

$$\Rightarrow -\frac{\pi}{3}\le\theta\le 0$$

$$\Rightarrow x\in\left[-\sqrt{3},0\right]$$

So, number of integral solutions is 2.

20. (c) : $a^2+b^2+c^2=4R^2(\sin^2A+\sin^2B+\sin^2C)$

$=2R^2(1-\cos 2A+1-\cos 2B+1-\cos 2C)$

$$=2R^2\left[3-\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\right]$$

$$=2R^2\left[3-\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)\right]$$

$$=2R^2\left[3-\left(\begin{array}{l}2\sin\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}+2\sin\frac{\pi}{7}\cdot\cos\frac{4\pi}{7}\\+2\sin\frac{\pi}{7}\cdot\cos\frac{6\pi}{7}\end{array}\right)\frac{1}{2\sin\frac{\pi}{7}}\right]$$

$$=2R^2\left[3-\frac{1}{2\sin\frac{\pi}{7}}\left(\begin{array}{r}\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}\\+\sin\frac{7\pi}{7}-\sin\frac{5\pi}{7}\end{array}\right)\right]$$

$$=2R^2\left[3+\frac{1}{2}\right]=7R^2$$

21. (d) : $\sin(\cot^{-1}(x+1))=\sin\left(\sin^{-1}\left(\dfrac{1}{\sqrt{x^2+2x+2}}\right)\right)$

$$\Rightarrow \sin(\cot^{-1}(x+1))=\frac{1}{\sqrt{x^2+2x+2}} \quad \ldots\text{(i)}$$

and $\cos(\tan^{-1}x)=\cos\left(\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)\right)=\dfrac{1}{\sqrt{1+x^2}}$ ...(ii)

By (i) and (ii), we get $\dfrac{1}{x^2+2x+2}=\dfrac{1}{1+x^2}$

22. (c) : $(x+3)^2+3+3\sin(a+bx)=0$

$x=-3$, $\sin(a+bx)=-1$

$\Rightarrow \sin(a-3b)=-1$

$$a-3b=(4n-1)\frac{\pi}{2}, n\in I$$

If $n=1 \Rightarrow a-3b=3\pi/2 \Rightarrow \cos(a-3b)=0$

23. (a) : Since, $\sum\cos A\operatorname{cosec}B\operatorname{cosec}C=\sum\dfrac{\cos A}{\sin B\sin C}$

$$=\frac{-\sum\cos(B+C)}{\sin B\sin C}$$

$$=\sum(1-\cot B\cot C)=3-\sum\cot A\cot B=2$$

24. (b) : $\tan^{-1}\left(\dfrac{\dfrac{a}{b+c}+\dfrac{b}{c+a}}{1-\dfrac{a}{b+c}\cdot\dfrac{b}{c+a}}\right)$ as $\dfrac{ab}{(b+c)(c+a)}<1$

But in right angled ΔABC

$$c^2=a^2+b^2 \therefore \tan^{-1}(1)=\frac{\pi}{4}$$

25. (a) : Given equations

$\Rightarrow 16+9+24\sin(P+Q)=37$

$$\Rightarrow P+Q=\frac{5\pi}{6} \text{ or } \frac{\pi}{6}$$

If $P+Q=\dfrac{\pi}{6}$ then $R=\dfrac{5\pi}{6}$

If $P<\dfrac{\pi}{6}, 3\sin P<\dfrac{3}{2}$, then $3\sin P+4\cos Q<\dfrac{3}{2}+4<6$

$\therefore P+Q=\dfrac{\pi}{6}$ is not possible $\therefore R=\dfrac{\pi}{6}$

26. (a, b, c, d) : $\dfrac{\sin\theta}{\sin\phi}\dfrac{\sin\theta}{\sin\phi}=\dfrac{\sin\theta}{\sin\phi}\dfrac{\cos\phi}{\cos\theta}$

$$\Rightarrow \frac{\sin\theta}{\sin\phi}=\frac{\cos\phi}{\cos\theta} \Rightarrow \sin 2\theta=\sin 2\phi$$

$$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta}=\frac{2\tan\phi}{1+\tan^2\phi}$$

$$\Rightarrow \frac{6\tan\phi}{1+9\tan^2\phi}=\frac{2\tan\phi}{1+\tan^2\phi}\Rightarrow\tan^2\phi=\frac{1}{3}\Rightarrow\tan\theta=\pm\sqrt{3}$$

27. (a, b, c, d) : (a) $\sin\frac{11\pi}{12}\sin\frac{5\pi}{12}$

$$=\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)=\frac{1}{2}\sin\left(\frac{\pi}{6}\right)=\frac{1}{4}\in Q$$

(b) $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right)=-\operatorname{cosec}\left(\frac{\pi}{10}\right)\sec\left(\frac{\pi}{5}\right)=-4\in Q$

(c) $1-2\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)=1-\frac{1}{4}=\frac{3}{4}\in Q$

(d) $\left(2\cos^2\frac{\pi}{9}\right)\left(2\cos^2\frac{2\pi}{9}\right)\left(2\cos^2\frac{4\pi}{9}\right)=\frac{1}{8}\in Q$

28. (a, c, d) : $x-y=\frac{1}{3}$ and

$\cos\{\pi(x+y)\}\cos\{\pi(x-y)\}=\frac{1}{2}$

$\Rightarrow\quad x+y=2n,\ n\in Z$

29. (a, b, c) : $2^{\operatorname{cosec}^2x}\sqrt{\frac{(y-1)^2+1}{2}}\le\sqrt{2}$

$\Rightarrow\quad 2^{\operatorname{cosec}^2x}\sqrt{(y-1)^2+1}\le 2$

$\Rightarrow\quad \operatorname{cosec}^2x=1$ and $y=1$

$\Rightarrow\quad x=\frac{\pi}{2},\frac{3\pi}{2},\ y=1$

30. (a, b, d) : α and β satisfy $x\cos\theta+y\sin\theta=2a$

$\Rightarrow\quad (x^2+y^2)\cos^2\theta-4ax\cos\theta+(4a^2-y^2)=0$

$$\cos\alpha+\cos\beta=\frac{4ax}{x^2+y^2},\ \cos\alpha\cdot\cos\beta=\frac{4a^2-y^2}{x^2+y^2}$$

$$2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)=1\ \Rightarrow\ 4\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\beta}{2}\right)=1$$

$\Rightarrow\quad \cos\alpha+\cos\beta=\cos\alpha\cdot\cos\beta$

$$\Rightarrow\quad \frac{4ax}{x^2+y^2}=\frac{4a^2-y^2}{x^2+y^2}\Rightarrow y^2=4a(a-x)$$

31. (a, b, c, d) : $\frac{k+1}{k-1}=2\cos 2A$

$$\frac{\sin 3A}{\sin A}=1+2\cos 2A=\frac{2k}{k-1}$$

Also $k=\frac{3-\tan^2A}{1-3\tan^2A}\Rightarrow k<\frac{1}{3},k>3$

32. (a, b, c, d) : $f_n(\theta)=\tan(\theta/2)\prod_{r=0}^{n}(1+\sec 2^r\theta)$

$$=\tan(\theta/2)\prod_{r=0}^{n}\left\{\frac{1+\cos(2^r\theta)}{\cos(2^r\theta)}\right\}$$

$$=\tan(\theta/2)\prod_{r=0}^{n}\frac{2\cos^2(2^{r-1}\theta)}{\cos(2^r\theta)}$$

$$=2^{n+1}\cdot\tan(\theta/2)\prod_{r=0}^{n}\frac{\cos^2(2^{r-1}\theta)}{\cos(2^r\theta)}$$

$$=2^{n+1}\cdot\tan(\theta/2)\cdot\cos^2(\theta/2)\prod_{r=0}^{n}\frac{\cos(2^{r-1}\theta)}{\cos(2^n\theta)}$$

$$=2^n\cdot\sin\theta\cdot\frac{\sin(2^n\theta)}{2^n.\sin\theta.\cos(2^n\theta)}=\tan(2^n\theta)$$

$\therefore$ Alternate (a) : $f_2\left(\frac{\pi}{16}\right)=\tan\left(\frac{\pi}{4}\right)=1$

Alternate (b) : $f_3\left(\frac{\pi}{32}\right)=\tan\left(\frac{\pi}{4}\right)=1$

Alternate (c) : $f_4\left(\frac{\pi}{64}\right)=\tan\left(\frac{\pi}{4}\right)=1$

Alternate (d) : $f_5\left(\frac{\pi}{128}\right)=\tan\left(\frac{\pi}{4}\right)=1$

33. (a, b) : $T_r=\sin^{-1}\left(\frac{\sqrt{r^2+2r}-\sqrt{r^2-1}}{r(r+1)}\right)$

$$T_r=\sin^{-1}\left(\frac{1}{r}\sqrt{1-\frac{1}{(r+1)^2}}-\frac{1}{r+1}\sqrt{1-\frac{1}{r^2}}\right)$$

$$T_r=\sin^{-1}\left(\frac{1}{r}\right)-\sin^{-1}\left(\frac{1}{r+1}\right)$$

$$\Rightarrow S_n=\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{n+1}\right)=\cos^{-1}\left(\frac{1}{n+1}\right)$$

34. (a, b) : $\sec\left(\frac{x+y}{2}\right)\le\frac{\sec x+\sec y}{2}$

35. (b) : $\sec\frac{\pi}{7},\sec\frac{3\pi}{7},\sec\frac{5\pi}{7}$ are the roots of

$x^3-4x^2-4x+8=0$

$$\sec\left(\frac{\pi}{7}\right)+\sec\left(\frac{3\pi}{7}\right)+\sec\left(\frac{5\pi}{7}\right)=4$$

$$8x^3-4x^2-4x+1=8\left(x-\cos\frac{\pi}{7}\right)\left(x-\cos\frac{3\pi}{7}\right)\left(x-\cos\frac{5\pi}{7}\right)$$

36. (b) : Put $x = 1$

$$\Rightarrow \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)=\frac{1}{8}$$

37. (d) : Put $x = -1$

$$\Rightarrow \cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)=\frac{\sqrt{7}}{8}$$

38. A - p; B - p; C - r; D - s

(A) $\tan^2\theta = \sin\theta\cos\theta \Rightarrow \sin\theta = \cos^3\theta$

$\therefore$ $(1 - \sin^2\theta) + (1 - 3\sin^2\theta) + 3\sin^4\theta - \sin^6\theta$

$= \cos^2\theta + (1 - \sin^2\theta)^3 = \cos^2\theta + \cos^6\theta = \cos^2\theta + \sin^2\theta = 1$

(B) $\sin 40° = \sin(60° - 20°)$

$$2\sin 20^0\cos 20^0=\frac{\sqrt{3}}{2}\cos 20^0-\frac{1}{2}\sin 20^0$$

$$4\cos 20^0=\sqrt{3}\cot 20^0-1$$

(C) $$\frac{3+\cot 76^0\cot 16^0}{\cot 76^0+\cot 16^0}=\frac{3\sin 76^0\sin 16^0+\cos 76^0\cos 16^0}{\sin(76^0+16^0)}$$

$$=\frac{2\sin 76^0\sin 16^0+\cos(76^0-16^0)}{\sin(76^0+16^0)}$$

$$=\frac{\cos 60^0-\cos 92^0+\cos 60^0}{\sin 92^0}=\frac{1-\cos 92^0}{\sin 92^0}=\tan 46^0$$

$= \cot 44°$

(D) $$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{2\pi}{18}\right)+....+\sin^2\left(\frac{\pi}{2}\right)=5$$

39. A - r; B - s; C - p; D - q

(A) Let $A=20^0 \Rightarrow 3A=60^0 \Rightarrow \cos 3A=\frac{1}{2}$

$\Rightarrow 4\cos^3 A-3\cos A=\frac{1}{2} \Rightarrow 8x^3 - 6x - 1 = 0,$

where $x = \cos 20^0$

(B) Let $A=10^0 \Rightarrow \sin 3A=\frac{1}{2} \Rightarrow 8x^3-6x+1=0,$

where $x = \sin 10^0$

(C) Let $A = 15° \Rightarrow 3A = 45°$

$$\Rightarrow \tan 3A = 1 \Rightarrow \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=1$$

$= x^3 - 3x^2 - 3x + 1 = 0$, where $x = \tan 15°$

(D) Let $A=6^0 \Rightarrow \sin 5A=\frac{1}{2} \Rightarrow 32x^5-40x^3+10x-1=0,$

where $x = \sin 6^0$

40. A - p; B - q; C - s; D - q, r

(A) Given, $\sin\theta = 3\sin(\theta + 2\alpha)$

$\Rightarrow \sin(\theta + \alpha - \alpha) = 3\sin(\theta + \alpha + \alpha)$

$\Rightarrow \sin(\theta + \alpha)\cos\alpha - \cos(\theta + \alpha)\sin\alpha$

$= 3\sin(\theta + \alpha)\cos\alpha + 3\cos(\theta + \alpha)\sin\alpha$

$\Rightarrow -2\sin(\theta + \alpha)\cos\alpha = 4\cos(\theta + \alpha)\sin\alpha$

$$\Rightarrow -\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}=\frac{2\sin\alpha}{\cos\alpha} \Rightarrow -\tan(\theta + \alpha) = 2\tan\alpha$$

$\Rightarrow \tan(\theta + \alpha) + 2\tan\alpha = 0$

(B) We have, $p\sin\theta + q\cos\theta = a$ ...(1)

and $p\cos\theta - q\sin\theta = b$ ...(2)

Squaring (1) and (2) and then adding, we get

$(p\sin\theta + q\cos\theta)^2 + (p\cos\theta - q\sin\theta)^2 = a^2 + b^2$

$\Rightarrow p^2(1) + q^2(1) - a^2 - b^2 = 0$

$\Rightarrow (p^2 - a^2) + (q^2 - b^2) = 0$

$\Rightarrow (p + a)(p - a) + (q + b)(q - b) = 0$

$$\Rightarrow \frac{p+a}{q+b}+\frac{q-b}{p-a}=0$$

(C) Consider, $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)$$

$$=\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)$$

$$=-\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

Also, $\cos\frac{10\pi}{7}=\cos\frac{4\pi}{7}$

So, $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7}-\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$$=2\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)=-\frac{1}{4}$$

(D) Clearly, $\sec\theta - \tan\theta = 1$

Also, 1 satisfy the given equation.

So, the roots of the given equation are 1 & $\sec\theta$.

41. (0) : Let $\theta=\frac{\pi}{18} \Rightarrow 6\theta=\frac{\pi}{3} \Rightarrow \cos 6\theta=\frac{1}{2}$

$$\Rightarrow 4\cos^3 2\theta-3\cos 2\theta=\frac{1}{2}$$

$$\Rightarrow 8(2\cos^2\theta-1)^3-6(2\cos^2\theta-1)=1.$$

Let $2\cos\theta = x$

$$\Rightarrow\ 8\left(2.\frac{x^2}{4}-1\right)^3-6\left(2.\frac{x^2}{4}-1\right)=1$$

$\Rightarrow\ (x^2 - 2)^3 - 3(x^2 - 2) = 1$

$\Rightarrow\ x^6 - 6x^4 + 9x^2 - 3 = 0$

$f'(x) = 6x\,(x^4 - 4x^2 + 3)$

$f'(x)=0 \Rightarrow x=0,\pm 1,\pm\sqrt{3}$

42. (0) : $\cos\theta\,(1 + \cos^2\theta) = \sin^2\theta$

$(1 - \sin^2\theta)\,(2 - \sin^2\theta)^2 = \sin^4\theta$

$\sin^6\theta = 4 - 8\sin^2\theta + 4\sin^4\theta$

On comparing with the given equation, we get

$a = 4,\ b = -8,\ c = 4$

$\therefore\ a + b + c = 0$

43. (0) : $\dfrac{\sin\theta}{\cos 3\theta}=\dfrac{2\sin\theta\cos\theta}{2\cos 3\theta\cos\theta}$

$$=\frac{\sin 2\theta}{2\cos 3\theta\cos\theta}=\frac{\sin(3\theta-\theta)}{2\cos 3\theta\cos\theta}$$

$$\therefore\ \frac{\sin\theta}{\cos 3\theta}=\frac{1}{2}[\tan 3\theta-\tan\theta] \quad \ldots(1)$$

$$\frac{\sin 3\theta}{\cos 9\theta}=\frac{1}{2}[\tan 9\theta-\tan 3\theta] \quad \ldots(2)$$

$$\frac{\sin 9\theta}{\cos 27\theta}=\frac{1}{2}[\tan 27\theta-\tan 9\theta] \quad \ldots(3)$$

$\therefore\ (1) + (2) + (3) \Rightarrow$

$$\frac{\sin\theta}{\cos 3\theta}+\frac{\sin 3\theta}{\cos 9\theta}+\frac{\sin 9\theta}{\cos 27\theta}=\frac{1}{2}[\tan 27\theta-\tan\theta]$$

Now, on comparing, we get

$A = 2,\ B = 27,\ C = 1$

$\therefore\ 27A - B - 27C = 0$

44. (3) : Let $\tan x = 2t$, $\tan y = 3t$, $\tan z = 5t$

$$\sum\tan x=\tan x\cdot\tan y\cdot\tan z \Rightarrow t^2=\frac{1}{3}$$

Now, $\tan^2 x + \tan^2 y + \tan^2 z = t^2\,(4 + 9 + 25)$

$= 38t^2,\ K = 3$

45. (4) : We have, $\sin\theta(1 + \sin^2\theta) = 1 - \sin^2\theta$

$\Rightarrow\ \sin\theta(2 - \cos^2\theta) = \cos^2\theta$

Squaring both sides, we get

$\sin^2\theta(2 - \cos^2\theta)^2 = \cos^4\theta$

$\Rightarrow\ (1 - \cos^2\theta)(4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$

$\Rightarrow\ -\cos^6\theta + 5\cos^4\theta - 8\cos^2\theta + 4 = \cos^4\theta$

$\therefore\ \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

46. (1) : $\alpha+2\alpha+4\alpha=7\alpha=\dfrac{\pi}{2}$

$$\Rightarrow\ \tan\frac{\pi}{2}=\tan(\alpha+2\alpha+4\alpha)$$

$$\Rightarrow \frac{1}{0}=\frac{\tan\alpha+\tan 2\alpha+\tan 4\alpha-\tan\alpha\cdot\tan 2\alpha\cdot\tan 4\alpha}{1-\tan\alpha\cdot\tan 2\alpha-\tan\alpha\cdot\tan 4\alpha-\tan 2\alpha\cdot\tan 4\alpha}$$

$\Rightarrow\ 1 - \tan\alpha\cdot\tan 2\alpha - \tan\alpha\cdot\tan 4\alpha - \tan 2\alpha\cdot\tan 4\alpha = 0$

$\Rightarrow\ \tan\alpha\cdot\tan 2\alpha + \tan 2\alpha\cdot\tan 4\alpha + \tan 4\alpha\cdot\tan\alpha = 1$

47. (2) : $r_1=\dfrac{\Delta}{s-a},\ r_2=\dfrac{\Delta}{s-b},\ r_3=\dfrac{\Delta}{s-c}$, Substitute this value and take abc common, we get

$$\text{L.H.S.}=\frac{abc}{\Delta}\left[\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}\right]=\frac{abc}{\Delta}\left[\sum\frac{s}{a}-3\right]$$

$\Rightarrow\ k = 2$

48. (1) : Let $A=\dfrac{\beta+\gamma-\alpha}{4}, B=\dfrac{\gamma+\alpha-\beta}{4}$ and

$$C=\frac{\alpha+\beta-\gamma}{4}$$

$\Rightarrow\ \tan A\ \tan B\ \tan C = 1$

$$\Rightarrow\ \frac{\sin A\sin B}{\cos A\cos B}=\frac{1}{\tan C}$$

$$\Rightarrow\ \frac{\sin A\sin B-\cos A\cos B}{\sin A\sin B+\cos A\cos B}=\frac{1-\tan C}{1+\tan C}$$

[By Componendo and Dividendo]

$$\Rightarrow\ \frac{-\cos(A+B)}{\cos(A-B)}=\frac{\sin\left(\frac{\pi}{4}-C\right)}{\cos\left(\frac{\pi}{4}-C\right)}$$

$$\Rightarrow 2\sin\left(\frac{\pi}{4}-C\right)\cos(A-B)+2\cos\left(\frac{\pi}{4}-C\right)\cos(A+B)=0$$

$$\Rightarrow \sin\left(\frac{\pi}{4}-C+A-B\right)+\sin\left(\frac{\pi}{4}-C-A+B\right)$$

$$+\cos\left(\frac{\pi}{4}-C+A+B\right)+\cos\left(\frac{\pi}{4}-C-A-B\right)=0 \quad \ldots(1)$$

$$A-B-C=\frac{\pi}{4}-\alpha,\ B-A-C=\frac{\pi}{4}-\beta,$$

$$C-A-B=\frac{\pi}{4}-\gamma \text{ and } A+B+C=\frac{\pi}{4}$$

$\therefore\ (1)\ \Rightarrow\ \cos\alpha + \cos\beta + \cos\gamma + 1 = 0$

Thus, $K = 1$.

ACE YOUR WAY CBSE XII

Series 8

LINEAR PROGRAMMING AND PROBABILITY

LINEAR PROGRAMMING

OPTIMISATION PROBLEM

A problem in which a linear function is to be optimised (maximise or minimise) satisfying certain linear inequalities is called optimisation problem.

LINEAR PROGRAMMING PROBLEM (L.P.P.)

Linear programming (L.P.) is an optimisation technique in which a linear function is optimised (*i.e.*, minimised or maximised) subject to certain constraints which are in the form of linear inequalities.

MATHEMATICAL FORMULATION OF L.P.P.

The procedure for mathematical formulation of a linear programming problem consists of the following steps :

Step I In every L.P.P. certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denoted them by $x_1, x_2, x_3, ...$

Step II Identify the objective function and express it as a linear function of the variables introduced in step I.

Step III In a L.P.P., the objective function may be in the form of maximising profits or minimising costs. So, after expressing the objective functions as a linear function of the decision variables, we must find the type of optimisation *i.e.,* maximisation or minimisation. Identify the type of the objective function.

Step IV Identify the set of the constraints, stated in terms of decision variables and express them as linear inequations or equations as the case may be.

USEFUL TERMS IN L.P.P.

Objective function	The linear function $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ which is to be optimised (maximised or minimised) is called the objective function.
Decision variables	The variables $x_1, x_2, x_3, ..., x_n$ whose values are to be decided, are called decision variables.
Linear constraints	The linear inequations (inequalities) or restrictions on the decision variables of a linear programming problem are called linear constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative constraints.
Convex region	A set S of points in a plane is said to be convex, if the line segment joining any two points in it, lies in it completely.
Feasible region	The set of points, whose co-ordinates satisfy the constraints of a linear programming problem, is said to be the feasible region.
Bounded and unbounded feasible region	A feasible region of a system of linear inequations is said to be bounded if it can be enclosed within a circle. But if the feasible region extends indefinitely in any direction, then the feasible region is called unbounded.
Corner points of a feasible region	Points in the region which is the intersection of two boundary lines are called corner points.

Feasible & infeasible solution	Points within and on the boundary of the feasible region of a L.P.P. represent feasible solutions. Any point outside the feasible region is called infeasible solution.
Optimal (feasible) solution	The optimal solution of the feasible region is the optimal value (maximum or minimum) of the function at any point.

THEOREMS

- Let R be the feasible region (convex polygon) for a linear programming problem and let $Z= ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at the corner point (vertex) of the feasible region.
- Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and minimum value on R and each of these occurs at a corner point (vertex) of R.

  Note : If R is unbounded, then a maximum or minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.

GRAPHICAL METHODS OF SOLVING L.P.P.

The graphical methods for solving L.P.P. is applicable to those problems which involve only two variables.

Corner Point Method

To solve a linear programming problem by corner point method, we follow the following steps :

(i) Formulate the given L.P.P. in mathematical form.

(ii) Convert all inequations into equations and draw their graphs.

(iii) Determine the region represented by each inequation.

(iv) Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the L.P.P.

(v) Determine the coordinates of the vertices (corner points) of the convex polygon obtained in step (ii). These vertices are known as the extreme points of the set of all feasible solutions of the L.P.P.

(vi) Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given L.P.P.

DIFFERENT TYPES OF L.P.P.

A few important linear programming problems are listed below :

- **Manufacturing problems :** In this problem, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.
- **Diet problems :** In these problems, we determine the amount of different kinds of constituents/ nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrients.
- **Transportation problems :** In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

PROBABILITY

CONDITIONAL PROBABILITY

If two events A and B are associated with the same random experiment, the conditional probability $P(A|B)$ of the occurrence of A knowing that B has occurred is given by

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)},P(B)\neq 0$$

Similarly, $P(B\,|\,A)=\dfrac{P(A\cap B)}{P(A)},P(A)\neq 0\cdot$

Properties of Conditional Probability

- If E_1 and E_2 are any two events associated with an experiment and $P(E_2) \neq 0$, then $0 \leq P(E_1|E_2) \leq 1$.
- If E is any event associated with an experiment, then $P(E|E) = P(S|E) = 1$, S being the sample space.

- If E, E_1 and E_2 are events associated with an experiment and $P(E) \neq 0$, then
  $P((E_1 \cup E_2)|E) = P(E_1|E) + P(E_2|E) - P((E_1 \cap E_2)|E)$
  In particular, if E_1 and E_2 are mutually exclusive, then
  $P((E_1 \cup E_2)|E) = P(E_1|E) + P(E_2|E)$.
- If E_1 and E_2 are any two events associated with an experiment, then $P(E_1^c|E_2) = 1 - P(E_1|E_2)$.

MULTIPLICATION THEOREM ON PROBABILITY

If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A) \cdot P(B|A) \ ; P(A) \neq 0$$

or $$P(A \cap B) = P(B) \cdot P(A|B) \ ; P(B) \neq 0$$

If $A_1, A_2, A_3, ..., A_n$ are n events related to a random experiment, then

$$P(A_1 \cap A_2 \cap .... \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|(A_1 \cap A_2)) \times .... \times P(A_n|(A_1 \cap A_2 \cap ... \cap A_{n-1})),$$

where $P(A_n|(A_1 \cap A_2 \cap ..... \cap A_{n-1}))$ represents the conditional probability of the event A_n, given that the events $A_1, A_2, ..., A_{n-1}$ have happened.

INDEPENDENT EVENTS

Two random experiments are said to be independent iff the probability of occurrence or non-occurrence of any event E_2 associated with the second experiment is independent of the outcome of the first experiment. This means that if E_1 is any event associated with the first experiment, then

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2)$$

Note :

- Two events E and F are said to be dependent if they are not independent, *i.e.*, if
  $P(E \cap F) \neq P(E) \cdot P(F)$
- Three events A, B and C are said to be mutually independent, if
  $$P(A \cap B) = P(A) \cdot P(B)$$
  $$P(A \cap C) = P(A) \cdot P(C)$$
  $$P(B \cap C) = P(B) \cdot P(C)$$
  and $$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$
  If at least one of the above is not true for three given events, we say that the events are not independent.

THEOREM OF TOTAL PROBABILITY

Partition of a sample space : A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(i) $E_i \cap E_j = \phi$, $i \neq j$, $i, j = 1, 2, 3 ....., n$

(ii) $E_1 \cup E_2 \cup ..... \cup E_n = S$ and

(iii) $P(E_i) > 0$ for all $i = 1, 2, ..., n$.

where the events, $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have non zero probabilities. For example any non empty event E and its complement E' form a partition of the sample space S since they satisfy $E \cap E' = \phi$ and $E \cup E' = S$.

Law of total probability : Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event of non-zero probability which occurs with E_1 or E_2 or ..... or E_n, then

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + ... + P(E_n)P(A|E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A|E_i)$$

BAYES' THEOREM

Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event of non- zero probability which occurs with E_1 or E_2 or ..... or E_n, then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{i=1}^{n} P(E_i)P(A|E_i)}, i = 1, 2, ..., n$$

RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTION

- **Random variable :** Random variable is simply a variable whose values are determined by the outcomes of a random experiment; generally it is denoted by capital letters such as X, Y, Z etc. and their values are denoted by the corresponding small letters x, y, z etc.
- **Probability distribution :** The system consisting of a random variable X along with $P(X)$, is called the probability distribution of X.
  The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	.....	x_n
$P(X)$	:	p_1	p_2	.....	p_n

  where $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2, ..., n$

- **Mean of a Random Variable**
  The mean of a random variable X is also called the expectation of X. *i.e.*, denoted by $E(X)$.
  So, $E(X) = \mu = p_1x_1 + p_2x_2 + ... + p_nx_n$ or $\sum_{i=1}^{n} p_ix_i$

- **Variance of a Random Variable**

$$\text{Var}(X)=\sum_{i=1}^{n}(x_i-\mu)^2\,p_i$$

OR

$$\text{Var}(X)=\sum_{i=1}^{n}p_i x_i^{\,2}-\left(\sum_{i=1}^{n}p_i x_i\right)^2=E(X^2)-[E(X)]^2$$

$$\sigma_x=\sqrt{\text{Var}(X)}=\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2\,p_i}$$

is called the Standard Deviation of the random variable X.

BERNOULLI TRIALS AND BINOMIAL DISTRIBUTION

- **Bernoulli trials :** A sequence of independent trials which can result in one of the two mutually exclusive possibilities success or failure such that the probability of success or failure in each trial is constant, then such repeated independent trials are called Bernoullian trials.
- **Binomial distribution :** A random variable X taking values 0, 1, 2, ..., n is said to have a binomial distribution with parameters n and p, if its probability distribution is given by

$$P(X=r)={}^nC_r\,p^r q^{n-r}$$

where p represents probability of success and q represents probability of non success, or failure and n is the number of trials.

Note : While using above probability density functions of the binomial distribution in solving any problem we should first of all examine whether all the conditions, given below are satisfied :

(i) There should be a finite number of trials.
(ii) The trials are independent.
(iii) Each trial has exactly two outcomes : success or failure.
(iv) The probability of an outcome remains the same in each trial.

- **Mean, variance and standard deviation :**
  - **(i)** Mean = np
  - **(ii)** Variance = npq
  - **(iii)** Standard deviation = $\sqrt{npq}$

Very Short Answer Type

1. A furniture dealer deals in only two items *viz.*, tables and chairs. He has ₹10, 000 to invest and a space to store at most 60 pieces. A table costs him ₹ 500 and chair ₹ 200. He can sell a table at a profit of ₹ 50 and a chair at a profit of ₹ 15. Assume that he can sell all the items that he buys. Formulate this problem as an L.P.P., so that he can maximise the profit.
2. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability of drawing two aces.
3. Six coins are tossed simultaneously. Find the probability of getting 3 heads.
4. If A and B are two independent events such that $P(A\cup B)=0.7$ and $P(A)=0.4$. Find $P(B)$.
5. A pair of dice is thrown 200 times. If getting a sum of 9 is considered a success, find the variance of the number of successes.

Short Answer Type

6. Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag I.
7. An instructor has a test bank consisting of 200 easy true/false questions, 100 difficult true/false questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from test bank, what is the probability that it will be a difficult question given that it is a multiple choice question?
8. Solve the following L.P.P. graphically :
   Minimize $Z=3x+5y$ subject to
   $-2x+y\le 4,\ x+y\ge 3,\ x-2y\le 2,\ x,y\ge 0$
9. A speaks truth in 60% of the cases and B in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact.
10. A dice is thrown 6 times. If 'getting an even number' is a success, what is the probability of
    (i) 5 successes (ii) at most 5 successes

Long Answer Type

11. Find the probability distribution of the number of green balls drawn when 3 balls are drawn, one by one, without replacement from a bag containing 3 green and 5 white balls.
12. Anil wants to invest at most ₹12,000 in Bonds A and B. According to the rules, he has to invest at least ₹2,000 in Bond A and at least ₹4,000 in Bond B. If the rate of interest on Bond A is 8% per annum

and on Bond B is 10% per annum, how should he invest his money for maximum interest? Solve the problem graphically.

13. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both spades. Find the probability of the lost card being a spade.

14. There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two cards drawn. Find the mean and variance of X.

15. A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three buildings, P, Q and R, for 15,000, 20,000 and 15,000 bricks respectively. The cost of transporting 1000 bricks to the builders from the depots (in rupees) are given below :

To \ From	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum?

SOLUTIONS

1. Let x and y be the number of tables and chairs respectively.
We have, $500x + 200y \le 10000$
Thus, the mathematical formulation of the L.P.P. is
Maximize $Z = 50x + 15y$ subject to the constraints
$$5x + 2y \le 100,\ x + y \le 60,\ x \ge 0, y \ge 0$$

2. $p = \frac{4}{52} = \frac{1}{13}$

$\therefore$ Required probability $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$

3. Let p be the probability of getting a head in the toss of a coin.

Then, $p = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2}$

Let X = No. of successes in the experiment. Then
$P(X = r) = {}^nC_r p^r \cdot q^{n-r}$

$$\therefore \quad P(X=3) = {}^6C_3\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{6-3} = \frac{20}{2^6} = \frac{20}{64} = \frac{5}{16}$$
[$\because n = 6$]

4. We know, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A)P(B)$
[$\because A$ and B are independent events]
$\Rightarrow 0.7 = 0.4 + P(B) - 0.4\,P(B)$
$\Rightarrow P(B) = \frac{0.3}{0.6} = \frac{1}{2} = 0.5$

5. p = getting a sum of 9 on a pair of dice $= \frac{4}{36} = \frac{1}{9}$
$\Rightarrow q = 1 - \frac{1}{9} = \frac{8}{9}$. Here $n = 200$

Variance $= npq = 200 \times \frac{1}{9} \times \frac{8}{9} = \frac{1600}{81}$

6. Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A|E_1) = P$ (drawing a red ball from bag I) $= \frac{3}{7}$

and $P(A|E_2) = P$(drawing a red ball from bag II) $= \frac{5}{11}$

Now, the probability of drawing a ball from bag I, being given that it is red, is $P(E_1|A)$. By using Bayes' theorem, we have

$$P(E_1 | A) = \frac{P(E_1)P(A | E_1)}{P(E_1)P(A | E_1) + P(E_2)P(A | E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{33}{68}$$

7. The given data may be tabulated as

	Easy	Difficult	Total
True/False	200	100	300
Multiple choice	500	400	900
Total	700	500	1200

Let us denote E = Easy questions, D = Difficult questions, T = True/False questions and M = Multiple choice questions
Number of difficult multiple choice questions = 400
Total number of questions = 1200
$P(D \cap M)$ = Probability of selecting a difficult and multiple choice question
$= \frac{400}{1200}$
Total number of multiple choice questions
$= 500 + 400 = 900$

$P(M)$ = Probability of selecting one multiple choice question

$$= \frac{900}{1200}$$

$$\therefore \quad P(D|M) = \frac{P(D \cap M)}{P(M)} = \frac{400}{1200} \div \frac{900}{1200} = \frac{4}{9}$$

8. First we draw the lines $-2x + y = 4$, $x + y = 3$ and $x - 2y = 2$

The feasible region has been shown shaded which is unbounded. The vertices of the feasible region are :

$$P\left(\frac{8}{3}, \frac{1}{3}\right), D\ (0, 3),\ B\ (0, 4)$$

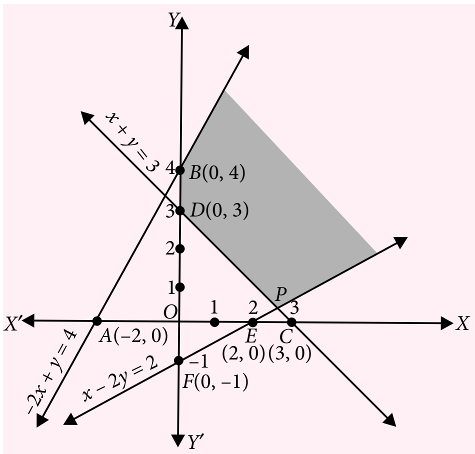
Given, $Z = 3x + 5y$

at $P\left(\frac{8}{3}, \frac{1}{3}\right)$, $Z = 3 \times \frac{8}{3} + 5 \times \frac{1}{3} = \frac{29}{3} = m(\text{minimum})$

at $D(0, 3)$, $\quad Z = 0 + 5 \times 3 = 15$

at $B(0, 4)$, $\quad Z = 0 + 5 \times 4 = 20$

Minimum value of $Z = \frac{29}{3}$ at $P\left(\frac{8}{3}, \frac{1}{3}\right)$ *i.e.*, when $x = \frac{8}{3}$ and $y = \frac{1}{3}$



9. Here the random experiment is "The stating of the fact by A and B".

Let $\ E$ = the event of A speaking the truth

and $\ F$ = the event of B speaking the truth

Then, $P(E) = \frac{60}{100} = \frac{3}{5}$ and $P(F) = \frac{90}{100} = \frac{9}{10}$

Required probability, P (A and B contradicting each other)

$$= P(E\bar{F} \text{ or } \bar{E}F) = P(E\bar{F}) + P(\bar{E}F)$$

$$= P(E) \cdot P(\bar{F}) + P(\bar{E}) \cdot P(F)$$

$$= P(E) \cdot [1 - P(F)] + [1 - P(E)] \cdot P(F)$$

$$= \frac{3}{5}\left(1 - \frac{9}{10}\right) + \left(1 - \frac{3}{5}\right) \cdot \frac{9}{10} = \frac{21}{50}$$

$\therefore$ A and B are likely to contradict each other in 42% cases.

10. We have,

the probability of success = the probability of even number on the die

$$= \frac{3}{6} = \frac{1}{2}$$

(i) The probability of getting 5 successes

$$P(5) = {}^6C_5 \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^5 = 6 \cdot \left(\frac{1}{2}\right)^6 = \frac{6}{64} = \frac{3}{32}$$

(ii) Probability of at most 5 successes $= 1 - P(6)$

$$= 1 - \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}$$

11. Let X denote the total number of green balls drawn in three draws without replacement. Clearly, there may be all green, 2 green, 1 green or no green at all. Thus, X can assume values 0, 1, 2 and 3. Let G_i denote the event of getting a green ball in i^{th} draw.

Now,

$P(X = 0)$ = Probability of getting no green ball in three draws

$$\Rightarrow P(X = 0) = P(\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3)$$

$$\Rightarrow P(X = 0) = P(\bar{G}_1)P(\bar{G}_2 | \bar{G}_1)P(\bar{G}_3 | (\bar{G}_1 \cap \bar{G}_2))$$

$$\Rightarrow P(X = 0) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5}{28}$$

$P(X = 1)$ = Probability of getting one green ball in three draws

$$\Rightarrow P(X = 1) = P\big((G_1 \cap \bar{G}_2 \cap \bar{G}_3) \cup (\bar{G}_1 \cap G_2 \cap \bar{G}_3) \cup (\bar{G}_1 \cap \bar{G}_2 \cap G_3)\big)$$

$$\Rightarrow P(X = 1) = P(G_1 \cap \bar{G}_2 \cap \bar{G}_3) + P(\bar{G}_1 \cap G_2 \cap \bar{G}_3) + P(\bar{G}_1 \cap \bar{G}_2 \cap G_3)$$

$$\Rightarrow P(X = 1) = P(G_1)P(\bar{G}_2 | G_1)P(\bar{G}_3 | (G_1 \cap \bar{G}_2)) + P(\bar{G}_1)P(G_2 | \bar{G}_1)P(\bar{G}_3 | (\bar{G}_1 \cap G_2)) + P(\bar{G}_1)P(\bar{G}_2 | \bar{G}_1)P(G_3 | (\bar{G}_1 \cap \bar{G}_2))$$

$$\Rightarrow P(X = 1) = \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{15}{28}$$

$P(X = 2)$ = Probability of getting two green balls in the three draws.

$$P(X = 2) = P\big((G_1 \cap G_2 \cap \bar{G}_3) \cap (\bar{G}_1 \cap G_2 \cap G_3) \cup (G_1 \cap \bar{G}_2 \cap G_3)\big)$$

$$\Rightarrow P(X = 2) = P(G_1)P(G_2 | G_1)P(\bar{G}_3 | (G_1 \cap G_2)) + P(\bar{G}_1)P(G_2 | \bar{G}_1)P(G_3 | (\bar{G}_1 \cap G_2)) + P(G_1)P(\bar{G}_2 | G_1)P(G_3 | (G_1 \cap \bar{G}_2))$$

$$\Rightarrow \quad P(X=2)=\frac{3}{8}\times\frac{2}{7}\times\frac{5}{6}+\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}=\frac{15}{56}$$

and

$P(X=3)$ = Probability of getting three green balls in the three draws.

$$P(X=3) = P(G_1 \cap G_2 \cap G_3)$$
$$= P(G_1)P(G_2|G_1)P(G_3|G_1 \cap G_2)$$

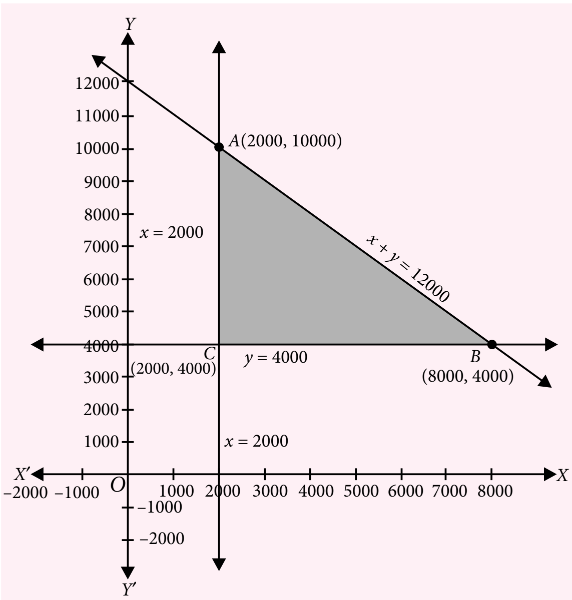
$$\Rightarrow \quad P(X=3)=\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}=\frac{1}{56}$$

Thus, the probability distribution of the number of green balls is given by

X :	0	1	2	3
$P(X)$:	$\frac{5}{28}$	$\frac{15}{28}$	$\frac{15}{56}$	$\frac{1}{56}$

12. Let Anil invest ₹ x in bond A and ₹ y in bond B, then

Interest on bond $A = ₹\, x \times \frac{8}{100} = ₹\, \frac{2x}{25}$



and interest on bond $B = ₹\, y \times \frac{10}{100} = ₹\, \frac{y}{10}$

His total annual interest $= ₹\left(\frac{2x}{25}+\frac{y}{10}\right)$

Thus, our L.P.P. is to

Maximise $Z=\frac{2x}{25}+\frac{y}{10}$ ... (i)

Subject to constraints

$x \geq 2000$ ... (ii)

$y \geq 4000$ ... (iii)

and $x + y \leq 12000$ ... (iv)

On plotting inequalities (ii) to (iv), we have the required region shown shaded in the figure.

Now, we evaluate Z at the corner points $A(2000, 10000)$, $B(8000, 4000)$ and $C(2000, 4000)$.

Corner Point	$Z=\frac{2x}{25}+\frac{y}{10}$	
$A(2000, 10000)$	1160 $\leftarrow$	Maximum
$B(8000, 4000)$	1040	
$C(2000, 4000)$	560	

$\therefore$ Z is a maximum *i.e.*, ₹1160 when $x = 2000$ and $y = 10000$.

So, Anil should invest ₹2000 in Bond A and ₹10000 in Bond B to earn maximum profit of ₹ 1160.

13. Let E_1, E_2, E_3 and E_4 be the events of losing a card of spades, clubs, hearts and diamonds respectively.

Let E be the event of drawing 2 spade cards from the remaining 51 cards.

Now $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52}=\frac{1}{4}$

$P(E|E_1)$ = Probability of drawing 2 spade cards given that a card of spade is missing.

$$=\frac{^{12}C_2}{^{51}C_2}=\frac{12\times 11}{51\times 50}=\frac{22}{425}$$

$P(E|E_2) = P(E|E_3) = P(E|E_4)$

$$=\frac{^{13}C_2}{^{51}C_2}=\frac{13\times 12}{51\times 50}=\frac{26}{425}$$

$\therefore$ By Bayes' theorem,

$$P(E_1 \mid E)=\frac{P(E_1)\cdot P(E \mid E_1)}{\sum_{i=1}^{4} P(E_i)\cdot P(E \mid E_i)}$$

$$=\frac{\frac{1}{4}\times\frac{22}{425}}{\frac{1}{4}\times\frac{22}{425}+\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{26}{425}}$$

$$=\frac{22}{22+(3\times 26)}=\frac{22}{100}=0.22$$

14. In this case the sample space contains $5 \times 4 = 20$ sample points of the type (x, y), $x \neq y$, $x, y \in \{1, 2, 3, 4, 5\}$

$\therefore$ Sum of the numbers on the two cards can take values from 3 to 9.

$$P(X=3)=P(\{(1,2),(2,1)\})=\frac{2}{20}=\frac{1}{10},$$

$$P(X=4)=P(\{(1,3),(3,1)\})=\frac{2}{20}=\frac{1}{10},$$

$$P(X=5)=P(\{(1,4),(4,1),(2,3),(3,2)\})=\frac{4}{20}=\frac{1}{5},$$

$$P(X=6)=P(\{(1,5),(5,1),(2,4),(4,2)\})=\frac{4}{20}=\frac{1}{5},$$

$$P(X=7)=P(\{(2,5),(5,2),(3,4),(4,3)\})=\frac{4}{20}=\frac{1}{5},$$

$$P(X=8)=P(\{(3,5),(5,3)\})=\frac{2}{20}=\frac{1}{10} \text{ and}$$

$$P(X=9)=P(\{(4,5),(5,4)\})=\frac{2}{20}=\frac{1}{10}.$$

Thus, the probability distribution of X is as follows :

X	3	4	5	6	7	8	9
$P(X)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Hence mean = $\Sigma X\,P(X)$

$$=3\times\frac{1}{10}+4\times\frac{1}{10}+5\times\frac{1}{5}+6\times\frac{1}{5}+7\times\frac{1}{5}+8\times\frac{1}{10}+9\times\frac{1}{10}$$

$$=\frac{3+4+10+12+14+8+9}{10}=\frac{60}{10}=6$$

and variance = $\Sigma X^2P(X)$ – (mean)^2

$$=3^2\times\frac{1}{10}+4^2\times\frac{1}{10}+5^2\times\frac{1}{5}+6^2\times\frac{1}{5}+7^2\times\frac{1}{5}+8^2\times\frac{1}{10}+9^2\times\frac{1}{10}-(6)^2$$

$$=\frac{9+16+50+72+98+64+81}{10}-36$$

$$=\frac{390}{10}-36=39-36=3$$

15. Let x and y units be shipped to buildings P and Q respectively from depot A. Then, the units shipped to other buildings from two depots A and B are shown in the figure. Therefore, there is the following constraints to the problem :

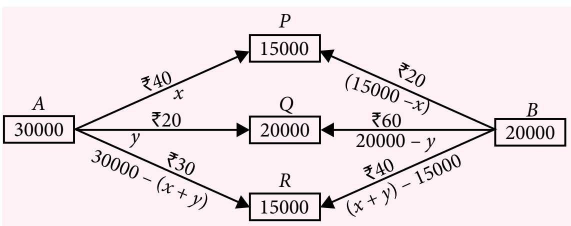
$x \geq 0$, $15000 - x \geq 0$, *i.e.*, $0 \leq x \leq 15000$ ...(1)

$y \geq 0$, $20000 - y \geq 0$, *i.e.*, $0 \leq y \leq 20000$ ...(2)

$30000 - (x + y) \geq 0$, $(x + y) - 15000 \geq 0$

i.e. $15000 \leq x + y \leq 30000$ ...(3)

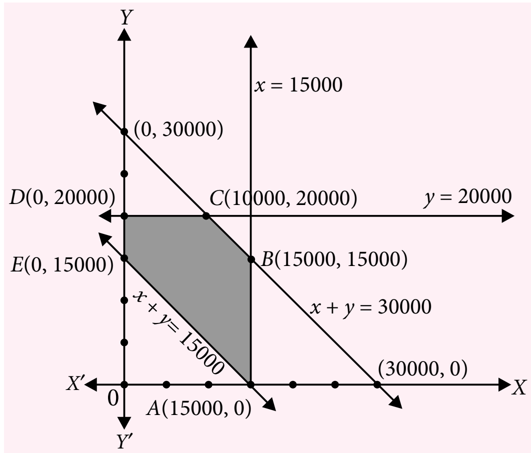
The total cost of transportation Z, for the distribution given in figure is



$$Z=\frac{1}{1000}(40x+20y+30[30000-(x+y)] + 20(15000 - x) + 60(20000 - y) + 40[(x + y) - 15000])$$

$$=\frac{1}{1000}(1800000+30x-30y)$$

Now, we have to minimise the objective function $Z=\frac{1}{1000}(1800000+30x-30y)$ (the cost of transportation) which satisfies all the constraints (1) to (3). The graph of inequations of the constraints given in (1) –(3) gives the feasible region $ABCDE$, where the vertices are A(15000, 0) B(15000, 15000), C(10000, 20000), D(0, 20000) and E(0, 15000).



The values of the objective function at the corner points are given in the following table :

Corner points (x, y)	Value of the objective function $Z=\frac{1}{1000}\times(1800000 + 30x - 30y)$
A(15000, 0)	Z = 2250
B(15000, 15000)	Z = 1800
C(10000, 20000)	Z = 1500
D(0, 20000)	Z = 1200
E(0, 15000)	Z = 1350

We find that minimum cost of transportation is ₹ 1350, when x = 0 units and y = 15000 units. Thus, the units transported from depot A and B to three buildings P, Q and R will be as follows so as to minimize the cost.

Depot	Bricks shipped to buildings			Total bricks
	P	Q	R	
A	0	20000	10000	30000
B	15000	0	5000	20000

WB JEE MOCK TEST PAPER

The entire syllabus of Mathematics of WB-JEE is being divided in to six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

Unit- I: Algebra, Unit-II: Trigonometry, Unit-III: Co-ordinate geometry of two dimension & three dimensions, Unit-IV: Calculus, Unit-V: Vector, Unit-VI: Statistics & probability.

CATEGORY 1

Category 1 shall have questions of 1 mark each. For incorrect response, 25% of full mark (1/4) would be deducted.

1. Given 5 line segments of the length 2, 3, 4, 5, 6 units. Then, the no. of triangles that can be formed by joining these segments is
(a) ${}^5C_3 - 3$ (b) ${}^5C_3$
(c) ${}^5C_3 - 1$ (d) ${}^5C_3 - 2$

2. How many numbers greater than 10,00,000 be formed from 2, 3, 0, 3, 4, 2, 3?
(a) 420 (b) 360 (c) 400 (d) 300

3. $\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right)-\sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$ is
(a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$

4. If in the expansion of $(1 + px)^n$, $n \in N$, then the coefficient of x and x^2 are 8 and 24 respectively, then
(a) $n = 3, p = 2$ (b) $n = 5, p = 3$
(c) $n = 4, p = 3$ (d) $n = 4, p = 2$

5. If $z=\frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$ then $|z|$ is equal to
(a) 1 (b) 3 (c) 0 (d) 2

6. If the 2nd and 5th terms of a G.P. are 24 and 3 respectively, then the sum of the first six terms is
(a) $\frac{189}{5}$ (b) $\frac{2}{189}$ (c) $\frac{189}{2}$ (d) $\frac{179}{2}$

7. If α and β are the roots of $x^2 - ax + b^2 = 0$, then $\alpha^2 + \beta^2$ is equal to
(a) $2a^2 - b^2$ (b) $a^2 + b^2$
(c) $a^2 - 2b^2$ (d) $a^2 - b^2$

8. If $\sin(\theta + \phi) = n\sin(\theta - \phi)$, $n \neq 1$, then the value of $\frac{\tan\theta}{\tan\phi}=$
(a) $\frac{n}{n-1}$ (b) $\frac{n+1}{n-1}$ (c) $\frac{n}{1-n}$ (d) $\frac{n-1}{n+1}$

9. If $\tan\theta = 5\tan\phi$ then the maximum value of $\tan^2(\theta - \phi)$ is
(a) $\frac{4}{5}$ (b) $\frac{4}{9}$ (c) $\frac{16}{25}$ (d) 1

10. $\frac{\sqrt{2}-\sin x-\cos x}{\sin x-\cos x}=$
(a) $\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$ (b) $\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$
(c) $\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)$ (d) $\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

11. $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=$
(a) e (b) e^{-1}
(c) 1 (d) none of these

12. Given that,
$$\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}=\frac{\pi}{2(a+b)(b+c)(c+a)},$$
then $\int_0^\infty \frac{dx}{(x^2+4^2)(x^2+9^2)}$ is

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(a) $\frac{\pi}{20}$ (b) $\frac{\pi}{40}$
(c) $\frac{\pi}{80}$ (d) none of these

13. The length of the perpendicular drawn from (1, 2, 3) to the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$ is
(a) 4 (b) 5 (c) 6 (d) 7

14. For any integer n, the integral
$\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx=$
(a) 1 (b) π
(c) 2π (d) none of these

15. If $[x]$ denotes the greatest integer $\le x$, then the value of $\lim_{x\to 0}|x|^{[\cos x]}$ is
(a) 0 (b) 1
(c) -1 (d) does not exists

16. If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side and perimeter is
(a) $\sqrt{3}:(2+\sqrt{3})$ (b) 1 : 6
(c) $1:(2+\sqrt{3})$ (d) 2 : 6

17. If $f(x)=\begin{cases}\frac{x^3+x^2-16x+20}{(x-2)^2}, & \text{if } x\ne 2\\ b, & \text{if } x=2\end{cases}$ is continuous for all x, then b is equal to
(a) 7 (b) 3 (c) 2 (d) 5

18. An integrating factor of the differential equation $y\log y\frac{dx}{dy}+x-\log y=0$ is
(a) log(log$y$) (b) log$y$
(c) $\frac{1}{\log y}$ (d) $\frac{1}{\log(\log y)}$

19. The area bounded by the curve $y=f(x)$, x-axis and the ordinates $x = l$ and $x = b$ is $(b - 1)\sin(3b + 4)$. Then $f(x)$ is
(a) $(x - 1)\cos(3x + 4)$
(b) $\sin(3x + 4)$
(c) $\sin(3x + 4) + 3(x - 1)\cos(3x + 4)$
(d) none of these

20. If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\log a$, then $\frac{dy}{dx}=$
(a) $\frac{y}{x}$ (b) $\frac{x}{y}$ (c) $\frac{x^2}{y^2}$ (d) $\frac{y^2}{x^2}$

21. If $y=e^{\sqrt{x}}+e^{-\sqrt{x}}$, then $\left(x\frac{d^2y}{dx^2}+\frac{1}{2}\frac{dy}{dx}\right)=$
(a) y (b) xy (c) $\frac{1}{4}y$ (c) $\sqrt{xy}$

22. If $I=\int\frac{dx}{\sqrt{(1-x)(x-2)}}$, then I is equal to
(a) $\sin^{-1}(2x - 3) + c$ (b) $\sin^{-1}(2x + 5) + c$
(c) $\sin^{-1}(3 - 2x) + c$ (d) $\sin^{-1}(5 - 2x) + c$

23. $\int\frac{\sin 2x}{(3+4\cos x)^3}dx=$
(a) $\frac{3\cos x+8}{(3+4\cos x)^2}+c$ (b) $\frac{3+8\cos x}{16(3+4\cos x)^2}+c$
(c) $\frac{3+\cos x}{(3+4\cos x)^2}+c$ (d) $\frac{3-8\cos x}{16(3+4\cos x)^2}+c$

24. If the algebraic sum of the deviations of 20 observations from 30 is 20, then the mean of observations is
(a) 30 (b) 30.1
(c) 29 (d) 31

25. The variance of first n natural numbers is
(a) $\frac{(n^2-1)}{12}$ (b) $\frac{(n^2-1)}{6}$
(c) $\frac{(n^2+1)}{6}$ (d) $\frac{(n^2+1)}{12}$

26. If $1, \log_{81}(3^x+48), \log_9\left(3^x-\frac{8}{3}\right)$ are in A.P., then the value of x equals
(a) 9 (b) 6 (c) 2 (d) 4

27. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their G.M., then $a : b$ is
(a) $6+\sqrt{7}:6-\sqrt{7}$ (b) $2+\sqrt{3}:2-\sqrt{3}$
(c) $5+\sqrt{6}:5-\sqrt{6}$ (d) none of these

28. If S, P and R are the sum, product and reciprocal of n terms of an increasing G.P. and $S^n = R^n \cdot P^k$, then k is equal to
(a) 1 (b) 2
(c) 3 (d) none of these

29. If the difference between the roots of $x^2+2px+q=0$ is two times the difference between the roots of $x^2+qx+\frac{p}{4}=0$, where $p\neq q$, then

(a) $p-q+1=0$ (b) $p-q-1=0$
(c) $p+q-1=0$ (d) $p+q+1=0$

30. The roots of the equation $\begin{vmatrix} 1+x & 3 & 5 \\ 2 & 2+x & 5 \\ 2 & 3 & x+4 \end{vmatrix}=0$ are

(a) 2, 1, −9 (b) 1, 1, −9
(c) −1, 1, −9 (d) −2, 1, −8

31. If $\begin{pmatrix} 0 & 3 & 2b \\ 2 & 0 & 1 \\ 4 & -1 & 6 \end{pmatrix}$ is singular, then the value of b is equal to

(a) −3 (b) 3 (c) −6 (d) 6

32. If A and B are square matrices of the same order and if $A=A^T$ and $B=B^T$, then $(ABA)^T=$

(a) BAB (b) ABA
(c) $ABAB$ (d) AB^T

33. The distance between the point (1, 2) and the point of intersection of the lines $2x+y=2$ and $x+2y=2$ is

(a) $\frac{\sqrt{17}}{3}$ (b) $\frac{\sqrt{16}}{3}$ (c) $\frac{\sqrt{17}}{5}$ (d) $\frac{\sqrt{19}}{3}$

34. $ABCD$ is a square with side a. If AB and AD are along the coordinate axes, then the equation of the circle passing through the vertices A, B and D is

(a) $x^2+y^2=\sqrt{2}a(x+y)$

(b) $x^2+y^2=\frac{a}{\sqrt{2}}(x+y)$

(c) $x^2+y^2=a(x+y)$

(d) $x^2+y^2=a^2(x+y)$

35. If the semi-major axis of an ellipse is 3 and the latus rectum is $\frac{16}{9}$, then the standard equation of the ellipse is

(a) $\frac{x^2}{9}+\frac{y^2}{8}=1$ (b) $\frac{x^2}{8}+\frac{y^2}{9}=1$

(c) $\frac{x^2}{9}+\frac{3y^2}{8}=1$ (d) $\frac{3x^2}{8}+\frac{y^2}{9}=1$

36. The one end of the latus rectum of the parabola $y^2-4x-2y-3=0$ is at

(a) (0, −1) (b) (0, 1)
(c) (0, −3) (d) (3, 0)

37. The eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{5}{4}$ and $2x+3y-6=0$ is a focal chord of the hyperbola, then the length of transverse axis is equal to

(a) $\frac{24}{5}$ (b) $\frac{5}{24}$ (c) $\frac{12}{5}$ (d) $\frac{6}{5}$

38. The solution of the differential equation $y'(y^2-x)=y$ is

(a) $y^3-3xy=C$ (b) $y^3+3xy=C$
(c) $x^3-3xy=C$ (d) $x^3-xy=C$

39. Let $\overrightarrow{OB}=\hat{i}+2\hat{j}+2\hat{k}$ and $\overrightarrow{OA}=4\hat{i}+2\hat{j}+2\hat{k}$. The distance of the point B from straight line passing through A and parallel to the vector $2\hat{i}+3\hat{j}+6\hat{k}$ is

(a) $\frac{7\sqrt{5}}{9}$ (b) $\frac{5\sqrt{7}}{9}$

(c) $\frac{3\sqrt{7}}{7}$ (d) $\frac{9\sqrt{5}}{7}$

40. The angle between the straight line $\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+s(\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}\times(2\hat{i}-\hat{j}+\hat{k})=4$ is

(a) $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ (b) $\sin^{-1}\left(\frac{\sqrt{2}}{6}\right)$

(c) $\sin^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (d) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$

41. Let A and B be two events. Then $1+P(A\cap B)-P(B)-P(A)$ is equal to

(a) $P(A^c\cup B^c)$ (b) $P(A\cap B^c)$
(c) $P(A^c\cap B)$ (d) $P(A^c\cap B^c)$

42. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle $\pi/3$ then the value of $|\vec{a}+\vec{b}|$ is

(a) equal to 1 (b) greater than 1
(c) equal to 0 (d) less than 1

43. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to

(a) 12 (b) 3 (c) 8 (d) 4

44. Let S be the set of real numbers. A relation R has been defined on S by $a\,R\,b \;\forall\; |a-b| \le 1$, then R is
(a) symmetric and transitive but not reflexive
(b) reflexive and transitive but not symmetric
(c) reflexive and symmetric but not transitive
(d) an equivalence relation

45. Let $f: N \to N$ defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

then f is
(a) onto but not one-one
(b) one-one and onto
(c) neither one-one nor onto
(d) one-one but not onto

46. The lines $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{-k}$ and $\dfrac{x-1}{k} = \dfrac{y-4}{2} = \dfrac{z-5}{1}$ are coplanar if
(a) $k = 2$ (b) $k = 0$
(c) $k = 3$ (d) $k = -1$

47. A fair die is rolled. Consider the events $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Then the conditional probability $P((A \cup B) \mid C)$ is
(a) 1/4 (b) 5/4 (c) 1/2 (d) 3/4

48. The set of values of x for which
$\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$ is
(a) ϕ (b) $\left\{\dfrac{\pi}{4}\right\}$
(c) $\left\{n\pi + \dfrac{\pi}{4}, n = 1, 2, 3\right\}$
(d) $\left\{2n\pi + \dfrac{\pi}{4}, n = 1, 2, 3\right\}$

49. $\displaystyle\int \frac{x^3 - 1}{x^3 + 1}\,dx =$
(a) $x - \log x + \log(x^2 + 1) - \tan^{-1}x + c$
(b) $x + \log x + \log(x^2 + 1) - \tan^{-1}x + c$
(c) $x + \log x + \dfrac{1}{2}\log(x^2 + 1) - \tan^{-1}x + c$
(d) none of these

50. If $x = a\cos^3\theta,\ y = a\sin^3\theta$, then $1 + \left(\dfrac{dy}{dx}\right)^2$ is
(a) $\tan^2\theta$ (b) 1
(c) $\tan\theta$ (d) $\sec^2\theta$

CATEGORY 2

Category 2 shall have questions of 2 marks each. For incorrect response, 25% of full mark (1/2) would be deducted.

51. If $a, b, c, d > 0$, $x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$ then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$ is equal to
(a) 1 (b) -1 (c) 0 (d) 2

52. The function $f(x) = \log(1 + x) - \dfrac{2x}{2 + x}$ is increasing in
(a) $(-1, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) none of these

53. $\displaystyle\int \left(\frac{\ln x - 1}{(\ln x)^2 + 1}\right)^2 dx =$
(a) $\dfrac{x}{x^2 + 1} + c$ (b) $\dfrac{\ln x}{(\ln x)^2 + 1} + c$
(c) $\dfrac{x}{(\ln x)^2 + 1} + c$ (d) none of these

54. The value of
$\left\{\dfrac{(81)^{\frac{1}{\log_5 9}} + 3^{\left(\frac{3}{\log_{\sqrt{6}} 3}\right)}}{409}\right\}\left\{(\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right\}$ is
(a) 0 (b) 1 (c) 2 (d) 3

55. $ABCD$ is a trapezium such that AB, DC are parallel and BC is perpendicular to them. If $\angle ADB = \dfrac{\pi}{4}$, $BC = 3$, $CD = 4$ then $AB =$
(a) $\dfrac{14}{3}$ (b) $\dfrac{7}{\sqrt{3}}$ (c) $\dfrac{13}{4}$ (d) $\dfrac{25}{7}$

56. The value of $\tan^{-1}\left(\cos\left(2\tan^{-1}\dfrac{3}{4}\right) + \sin\left(2\cot^{-1}\dfrac{1}{2}\right)\right)$ is
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $> \dfrac{\pi}{4}$ (d) $< \dfrac{\pi}{4}$

57. Let $z = \cos\theta + i\sin\theta$, then the value of
$\displaystyle\sum_{m=1}^{15} \mathrm{Im}(z^{2m-1})$ at $\theta = 2°$ is
(a) $\dfrac{1}{\sin 2°}$ (b) $\dfrac{1}{3\sin 2°}$
(c) $\dfrac{1}{2\sin 2°}$ (d) $\dfrac{1}{4\sin 2°}$

58. The equation of the plane containing the lines $2x - 5y + z = 3$, $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$ is
(a) $x + 3y + 6z = 1$ (b) $2x + 6y + 12z = -13$
(c) $2x + 6y + 12z = 13$
(d) $x + 3y + 6z = -7$

59. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R$ is a
(a) circle of radius $\sqrt{2}$
(b) circle of radius $\sqrt{3}$
(c) straight line parallel to x-axis
(d) straight line parallel to y-axis.

60. $\lim\limits_{x \to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x \tan 4x} =$
(a) 2 (b) 1/2 (c) 4 (d) 3

61. The area (in square units) of the region described by $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is
(a) $\dfrac{15}{64}$ (b) $\dfrac{9}{32}$ (c) $\dfrac{7}{32}$ (d) $\dfrac{5}{64}$

62. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls is
(a) $220\left(\dfrac{1}{3}\right)^{12}$ (b) $22\left(\dfrac{1}{3}\right)^{11}$
(c) $\dfrac{55}{3}\left(\dfrac{2}{3}\right)^{11}$ (d) $\dfrac{55}{3}\left(\dfrac{2}{3}\right)^{10}$

63. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 \le x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is
(a) 10/3 (b) 4 (c) 2 (d) 16/5

64. Let $f'(x) = \dfrac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\dfrac{1}{2}\right) = 0.$
If $m \le \int\limits_{1/2}^{1} f(x)dx \le M$ then the possible values of m and M are
(a) $m = 13, M = 24$ (b) $m = \dfrac{1}{4}, M = \dfrac{1}{2}$
(c) $m = -11, M = 0$ (d) $m = 1, M = 12$

65. If $\int\limits_{\ln 2}^{x} \dfrac{du}{\sqrt{e^u - 1}} = \dfrac{\pi}{6}$, then the value of x is
(a) 4 (b) ln8
(c) ln4 (d) none of these

CATEGORY 3

In category 3, questions may have more than one correct option. All correct answers only will yield two marks. Candidates must mark all the correct options in order to be awarded full marks

66. A line L is a tangent to the parabola $y^2 = 4x$ and a normal to the parabola $x^2 = \sqrt{2}y$. The distance of the origin from L is
(a) 0 (b) $\dfrac{2}{\sqrt{3}}$ (c) $\dfrac{3}{\sqrt{5}}$ (d) 2

67. Let $f(x) = \ln|x|$ and $g(x) = \sin x$. If A is the range of $f(g(x))$ and B is the range of $g(f(x))$, then
(a) $A \cup B = (-\infty, 1]$ (b) $A \cap B = (-\infty, \infty)$
(c) $A \cap B = [-1, 0]$ (d) $A \cap B = [0, 1]$

68. The general solution of the equation $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$ is
(a) $n\pi$ (b) $2n\pi$
(c) $2n\pi + \dfrac{\pi}{2}$ (d) $2n\pi + \dfrac{\pi}{4}$

69. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\dfrac{1}{2}, \dfrac{1}{4}$ and $\dfrac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is(are) true?
(a) $P(X_1^c \mid X) = \dfrac{3}{16}$
(b) P(exactly two engines of the ship are functioning $|X) = \dfrac{7}{8}$
(c) $P(X \mid X_2) = \dfrac{5}{16}$ (d) $P(X \mid X_1) = \dfrac{7}{16}$

70. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$ and $\vec{b} \cdot \vec{c} = 24$ then which of the following is true?
(a) $\dfrac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ (b) $\dfrac{|\vec{c}|^2}{2} + |\vec{a}| = 30$
(c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$
(d) $\vec{a} \cdot \vec{b} = -72$

Rvp

71. Let E_1 and E_2 be two ellipses whose centres are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2 respectively, then the correct expression(s) is (or)

(a) $e_1^2 + e_2^2 = \frac{43}{40}$ (b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(c) $|e_1^2 - e_2^2| = \frac{5}{8}$ (d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

72. If $f(x) = x\left(\frac{e^{|x|+[x]} - 2}{|x|+[x]}\right)$, then

(a) $\lim_{x\to 0^+} f(x) = -1$ (b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 0} f(x) = -1$ (d) $\lim_{x\to 0} f(x) = 0$

73. The value of x satisfying the equation

$$x^4 - 2\left(x\sin\left(\frac{\pi}{2}x\right)\right)^2 + 1 = 0 \text{ is}$$

(a) 1 (b) –1

(c) 0 (d) no value of x

74. Let $f(x)$ be real valued function such that $f(x+y) = f(x)f(a-x) + f(y) + f(a-x)\ \forall\ x, y \in R$ then for some real a

(a) $f(x)$ is a periodic function

(b) $f(x)$ is a constant function

(c) $f(x) = \frac{1}{2}$ (d) $f(x) = \frac{1}{2}\cos x$

75. System of equations $x + 3y + 2z = 6$, $x + \lambda y + 2z = 7$ and $x + 3y + 2z = \mu$ has

(a) unique solution if $\lambda = 2$, $\mu = 6$

(b) infinitely many solution, if $\lambda = 4$, $\mu = 6$

(c) no solution if $\lambda = 5$, $\mu = 7$

(d) no solution if $\lambda = 3$, $\mu = 5$

ANSWER KEYS

1. (a)	2. (b)	3. (d)	4. (d)	5. (d)
6. (c)	7. (c)	8. (b)	9. (a)	10. (a)
11. (b)	12. (d)	13. (d)	14. (d)	15. (b)
16. (a)	17. (a)	18. (b)	19. (c)	20. (a)
21. (c)	22. (a)	23. (b)	24. (d)	25. (a)
26. (c)	27. (b)	28. (b)	29. (d)	30. (b)
31. (c)	32. (b)	33. (a)	34. (c)	35. (c)
36. (a)	37. (a)	38. (a)	39. (d)	40. (a)
41. (d)	42. (b)	43. (b)	44. (c)	45. (a)
46. (b)	47. (d)	48. (a)	49. (d)	50. (d)
51. (c)	52. (a)	53. (c)	54. (b)	55. (d)
56. (c)	57. (d)	58. (a)	59. (a)	60. (a)
61. (b)	62. (c)	63. (c)	64. (d)	65. (c)
66. (a, b)		67. (a, c)		68. (b, c)
69. (b, d)		70. (a, c, d)		71. (a, b)
72. (a, b)		73. (a, b)		74. (a, b, c)
75. (b, c, d)				

MATHS MUSING

SOLUTION SET-156

1. **(d):** $\dfrac{2ab}{a+b} = 2016$

$\therefore$ $(a - 1008)(b - 1008) = (1008)^2 = 2^8 \cdot 3^4 \cdot 7^2$. The number of pairs $= \dfrac{1}{2}$ (The number of divisors – 1)

$$= \frac{9 \cdot 5 \cdot 3 - 1}{2} = 67$$

2. **(c) :** $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{2016}$

$(a - 2016)(b - 2016) = (2016)^2 = 2^{10} \cdot 3^4 \cdot 7^2$

Number of pairs $= \dfrac{1}{2}(11 \cdot 5 \cdot 3 - 1) = 82$

3. **(b):** $(5x - 2)^{2016} = (5x)^{2016} - 2016(5x)^{2015} \times 2$

$$+\frac{2016 \cdot 2015}{2}(5x)^{2014} \times 2^2 + \ldots = 0$$

The sum of the roots $= \dfrac{2016 \cdot 2015\,(5)^{2014} \times 2^2}{2 \cdot 2016\,(5)^{2015} \times 2}$

$= 403$

4. **(a) :** $(10^{2016} + 5)^2 = 225\,N$

$9N = (2 \cdot 10^{2015} + 1)^2 = 4 \cdot 10^{4030} + 4 \cdot 10^{2015} + 1$

$\therefore$ $N = 4\dfrac{(10^{4030} - 1)}{(10 - 1)} + 4\dfrac{(10^{2015} - 1)}{(10 - 1)} + 1$

$= 444...4 + 444...4 + 1$

$\therefore$ $S = (4 \times 4030) + (4 \times 2015) + 1 = 24181$, with digit sum 16.

5. **(b):** $\displaystyle\int_0^{2016} x^5(2016 - x)^6\,dx$ Put $x = 2016t$

$$I = 2016^{12}\int_0^1 t^5(1-t)^6\,dt = 2016^{11} \cdot 2016 \cdot \frac{5!6!}{12!}$$

$$= \frac{4}{11} \cdot 2016^{11}$$

6. **(a, b, c) :** In triangle ABC if $C = \dfrac{\pi}{2}$

$2r + c = a + b$, $(a + b - 2r)^2 = a^2 + b^2$

$(a - 2r)(b - 2r) = 2r^2 = 2(2016)^2 = 2^{11} \cdot 3^4 \cdot 7^2$

Number of triangles is $\dfrac{1}{2}$ (number of divisors of $2r^2$)

$= 12 \cdot 5 \cdot 3 = 2^2 \cdot 3^2 \cdot 5$

7. **(b):** $a^2 - b^2 = (a - b)(a + b)$ is divisible by 5 if both a and b are from any of the five rows, or one from row 1 and the other from row 4, or one from row 2 and the other from row 3.

1	6	11	16	21
2	7	12	17	22
3	8	13	18	23
4	9	14	19	24
5	10	15	20	25

$\therefore$ The number of pairs $(a, b) = 5\dbinom{5}{2} + 2\dbinom{5}{1}^2$

$= 100$

8. **(b):** $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$ is divisible by 5 if both a and b over from way of the five rows or one each from any of the first 4 rows.

$\therefore$ The number of pairs $(a, b) = 5\dbinom{5}{2} + \dbinom{4}{2}\dbinom{5}{1}^2$

$= 200$

9. **(1):** $a < b < c$, $a + b + c = 25$

a	b	c
4	5	16
	6	15
	⋮	
	10	11

a	b	c
5	6	14
	7	13
	8	12
	9	11

a	b	c
6	7	12
	8	11
	9	10

a	b	c
7	8	10

Total (a, b, c) are $6 + 4 + 3 + 1 = 14$ and taking the permutations $14 \times 3! = 84$.

$a + b + c = 25$, where $a, b, c \geq 4$

$\Rightarrow a + b + c = 16$, where $a, b, c \geq 1$. The number of positive integer solutions $= \dbinom{15}{2} = 105$

Prob. $= \dfrac{84}{105} = \dfrac{4}{5} = \dfrac{m}{n}$, $n - m = 1$

10. **(a) :** (P) $\to$ 3, (Q) $\to$ 5, (R) $\to$ 4, (S) $\to$ 1

(P) $3(4^2 + 5^2) - 40\sum \vec{a} \cdot \vec{b} \leq 123 + 60 = 183$

since $-2\sum \vec{a} \cdot \vec{b} \leq 3$

(Q) $t_1 + t_2 + t_3 = -1 - 1 + 8 = 6$. Sum $= 33 \times 6 - 1 = 197$

(R) Using Leibnitz rule : $y_4(1) = 1 \times 24 \times (-1)(-8) = 192$

(S) $(\alpha + \beta)(r + s) = -15 \Rightarrow p = 15$

Product of roots $= 158 = q$, $\therefore$ $p + q = 173$

Solution Sender of Maths Musing

SET-155

1. Khokon Kumar Nandi (W.B.)
2. Gajula Raninder (Karim Nagar (TS))

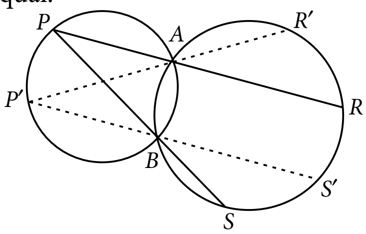
OLYMPIAD CORNER

1. We are considering triangles ABC in space.

(a) What conditions must be fulfilled by the angles α, β, γ of triangle ABC in order that there exists a point P in space such that $\angle APB$, $\angle BPC$, $\angle CPA$ are right angles?

(b) Let d be the maximum distance among PA, PB, PC and let h be the longest altitude of triangle ABC. Show that $(\sqrt{6}/3)h \le d \le h$.

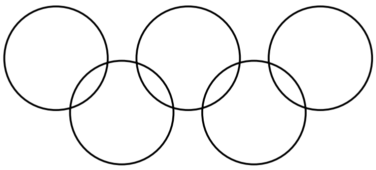
2. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



3. For any positive integer n, evaluate a_n/b_n, where

$$a_n = \sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}, b_n = \prod_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}.$$

4. There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



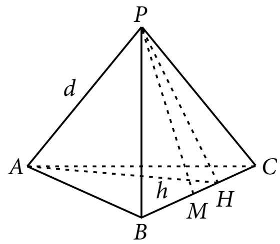
5. Given are real numbers $a_1, a_2, ..., a_n$ with $\sum_{i=1}^{n} a_i = 0$. Determine

$$\sum_{i=1}^{n} \frac{1}{a_i(a_i + a_{i+1})(a_i + a_{i+1} + a_{i+2})...(a_i + a_{i+1} + \cdots + a_{i+n-2})}$$

where $a_{n+1} = a_1$, $a_{n+2} = a_2$, etc., assuming that the denominators are non zero.

SOLUTIONS

1. (a) By Pythagoras theorem, we have $AB^2 = PA^2 + PB^2$, $AC^2 = PA^2 + PC^2$ and $BC^2 = PB^2 + PC^2$.



Hence $AB^2 + AC^2 - BC^2 = 2PA^2 > 0$, thus $\angle BAC < 90°$, *i.e.*, $\alpha < 90°$.

Similarly, we get $\beta < 90°$ and $\gamma < 90°$. Conversely, if α, β and γ are all acute angles, we may prove that there exists a point P such that $\angle APB$, $\angle ABC$, $\angle CPA$ are right angles.

(b) We may without loss of generality assume that $PA \ge PB \ge PC$, so $PA = d$, Because $AB^2 = AP^2 + BP^2 \ge AP^2 + CP^2 = AC^2$ and $AC^2 = AP^2 + CP^2 \ge BP^2 + CP^2 = BC^2$, we have $AB \ge AC \ge BC$.

Let H be the foot of the perpendicular from A to BC, then AH is the longest altitude of ΔABC, so $AH = h$. As $AP \perp BP$ and $AP \perp CP$, AP is perpendicular to the plane of BPC. Thus $AP \perp BC$ and $AP \perp PH$ so that $AP < AH$, *i.e.*, $d < h$. ...(1)

Because $AP \perp BP$ and $AH \perp BC$, we get BC is perpendicular to the plane of APH. Thus, we have

$BC \perp PH$.

Let M be the midpoint of BC, then $PH \le PM$. As $\angle BPC = 90°$, we have $PM = BM = MC = \frac{1}{2}BC$. Hence, $2PH \le 2PM = BC$, so that

$$4PH^2 \le BC^2 = PB^2 + PC^2 \le 2PA^2 \quad \text{...(2)}$$

As $\angle APH = 90°$, we get

$$PH^2 = AH^2 - AP^2 = h^2 - d^2 \quad \text{...(3)}$$

From (2) and (3), we have

$$4(h^2 - d^2) \le 2d^2, \text{ i.e., } 2h^2 \le 3d^2,$$

from which, we have

$$\frac{\sqrt{6}}{3}h \le d. \quad \text{...(4)}$$

From (1) and (4) we obtain $\frac{\sqrt{6}}{3}h \le d < h$, as required.

2. Because opposite angles of a cyclic quadrilateral are supplementary, we have that $\angle PBA = \pi - \angle ABS = \angle ARS$. Similarly $\angle PAB = \angle BSR$. Thus ΔPAB and ΔPSR are similar, from which

$$\frac{PA}{PS} = \frac{PB}{PR} = \frac{AB}{RS}$$

(Notice that this also gives the 'power of the point' result for P, $PA \cdot PR = PB \cdot PS$)

Similarly $\frac{P'A}{P'S} = \frac{P'B}{P'R'} = \frac{AB}{R'S'}$

Consider now traingles APS and $AP'S'$. We have $\angle APS = \angle APB = \angle AP'B = \angle AP'S'$, because P, P' lie on the same arc of chord AB of the one circle. From the fact that S and S' lie on the same arc of chord AB of the second circle $\angle AS'P' = \angle ASP$. But then ΔAPS and $\Delta AP'S'$ are similar. Thus

$$\frac{PA}{PS} = \frac{P'A}{P'S'}. \text{ So } \frac{AB}{RS} = \frac{AB}{R'S'}$$

Thus $RS = R'S'$ and the arcs are equal.

3. Using De Moivre's theorem

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n,$$

one finds easily that

$$\sin n\theta = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta$$

So

$$\sin(2n+1)\theta = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \cos^{2n-2k}\theta \sin^{2k+1}\theta$$

$$= \tan\theta \cos^{2n+1}\theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta.$$

Thus

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta = 0$$

for $\theta = \frac{j\pi}{2n+1}, 1 \le j \le n$

So $\tan^2 \frac{j\pi}{2n+1}, 1 \le j \le n,$ are the roots of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} x^k = 0$ and thus also of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{n-k} = 0.$ Since a_n and b_n are the sum and product of the roots, respectively, we have

$$a_n = \binom{2n+1}{2} = n(2n+1) \text{ and } b_n = \binom{2n+1}{2n} = 2n+1,$$

and so $\frac{a_n}{b_n} = n$

4. For the five rings, we have

$a+b=b+c+d=d+e+f=f+g+h$

$=h+i=N.$ ...(1)

Since we are dealing with the nine non-zero decimal digits, we have $\sum_{1}^{9} j = 9(10)/2 = 45.$ The five regions sum to a common N for $45/5 = 9$ but then one pair must be 9 + 0 or one triplet 9 + 0 + 0, which isn't allowed. So $N > 9$. Since $a + b = h + i$, there must be at least two pairs of decimal digits that sum to N. For $10 \le N \le 15$, we have

$N = 9 + a = 8 + (1 + a) = ...,$ for $1 \le a \le 6$

while $N = 16 = 9 + 17$ and $N = 17 = 9 + 8$ only. So $N \le 15$.

From (1)

$a+b=b+c+d$ or $a=c+d$ ...(2)

and

$h+i=f+g+h$ or $i=f+g$ ...(3)

The five central digits must equal $45 - 2N$

$(c + d) + e + (f + g) = a + e + i = 45 - 2N$

So, we have

N	$2N$	$45 - 2N$	a, e, i
10	20	25	9, 8 – no digit available
11	22	23	9, 8, 6;...
12	24	21	9, 8, 4;...
13	26	19	9, 8, 2;...
14	28	17	9, 7, 1;...
15	30	15	9, 5, 1;...

So $11 \le N \le 15$.

5. Let

$$S=\sum_{i=1}^{n}\frac{1}{a_i(a_i+a_{i+1})\ldots(a_i+a_{i+1}+\cdots+a_{i+n-2})}$$

$$=\sum_{i=1}^{n}\frac{1}{a_{i+1}(a_{i+1}+a_{i+2})\ldots(a_{i+1}+a_{i+2}+\cdots+a_{i+n-1})}$$

be the given sum. Let $b_i = \sum_{r=1}^{i} a_r$ so $b_n = 0$. Also, let $p=\prod_{r=1}^{n-1} b_r$. Since for all $j \ne i$

$$b_j - b_i = b_n - b_i + b_j = a_{i+1} + a_{i+2} + \cdots + a_n + a_1 + a_2 + \cdots + a_j$$
$$= a_{i+1} + a_{i+2} + \cdots + a_n + a_{n+1} + \cdots + a_{n+j},$$

We have

$$S=\sum_{i=1}^{n}\prod_{\substack{1\le j\le n\\ j\ne i}}\frac{1}{b_j-b_i}=\frac{1}{p}+\sum_{i=1}^{n-1}\prod_{\substack{1\le j\le n\\ j\ne i}}\frac{1}{b_j-b_i}$$

$$=\frac{1}{p}+\sum_{i=1}^{n-1}\left(-\frac{1}{b_i}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{1}{b_j-b_i}\right).$$

Using Lagrange's interpolation, we let

$$F(x)=\sum_{i=1}^{n-1}\left(-\frac{1}{p}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{b_j-x}{b_j-b_i}\right)$$

So that $F(b_k) = -1/p$ for all $1 \le k \le n - 1$. Note that $F(x)$ is a polynomial in x of degree at most $n - 2$, yet it is a constant at $n - 1$ distinct values. Hence $F(x)$ is a constant, $F(x) = -1/p$. Since

$$-\frac{1}{p}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{b_j}{b_j-b_i}=-\frac{1}{b_i}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{1}{b_j-b_i}$$

we get $F(0) = S - 1/p$ and thus

$$S=F(0)+\frac{1}{p}=0.$$

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 \cdot A_3 \ldots\ldots A_n =$

(a) $\dfrac{1}{n(n+1)}$ (b) $\dfrac{1}{2^n n(n+1)}$

(c) $\dfrac{1}{2^{n-1} n(n+1)}$ (d) $\dfrac{1}{2^{n-2} n(n+1)}$

2. The area bounded by $y = \cos^{-1}(\cos x)$, x-axis and the lines $x = 0$, $x = 2\pi$ is

(a) π^2 (b) $2\pi^2$

(c) $4\pi^2$ (d) Area is not bounded

3. The area bounded by $y = x^2$, the lines $y = 1$ and $y = 2$, on the left of y–axis is

(a) $\frac{4}{3}\left(2\sqrt{2}-1\right)$ (b) $\frac{4}{3}\left(1-2\sqrt{2}\right)$

(c) $\frac{2}{3}\left(2\sqrt{2}-1\right)$ (d) $\frac{2}{3}\left(1-2\sqrt{2}\right)$

4. If $\displaystyle\int \frac{\cos^4 x+1}{\cot x-\tan x}dx =$ $a\cos^2 2x + b\cos 2x + c \ln|\cos 2x| + \lambda$, λ being a constant of integration, then

(a) $a=-\frac{1}{8}$ (b) $b=-\frac{1}{8}$

(c) $c=\frac{5}{8}$ (d) $a + b + c = 0$

5. $\displaystyle\int \frac{dx}{x(1+x^4)} =$

(a) $3x^2 + c$ (b) $\ln\dfrac{x}{1+x^2}+c$

(c) $\ln|x|-\frac{1}{4}\ln(1+x^4)+c$

(d) None of these

6. Let $f(x)$ be a polynomial of degree three such that $f(0) = 1$, $f(1) = 2$ and 0 is a critical point of f having no local extreme. Find $\displaystyle\int \frac{f(x)dx}{\sqrt{x^2+7}}$.

7. Evaluate $\displaystyle\int \sin^{-1}\left(2x\sqrt{1-x^2}\right)dx,\ x\in\left[\frac{1}{\sqrt{2}},1\right]$.

8. Using derivatives up to order of $f(x) = x^{1/3} + x^{2/3}$ and giving main points, draw the graph of $y = f(x)$. Also find the area bounded by the curve $y = f(x)$ and the x – axis.

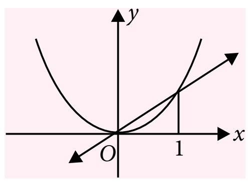
9. Solve $\dfrac{d^3y}{dx^3}-8\dfrac{d^2y}{dx^2}=0$, given that $y(0)=\frac{1}{8}$, $y'(0)=0$ and $y''(0)=1$.

10. Draw the graph of the function f defined by $f(x) = \cot^{-1}x$, giving main points. Hence or otherwise evaluate $\displaystyle\int_{-\sqrt{3}}^{1} [\cot^{-1} x]dx$ ([.] denotes the greatest integer function).

SOLUTIONS

1. (d) : $A_n=\displaystyle\int_0^1 (x-x^n)dx=\left[\frac{x^2}{2}-\frac{x^{n+1}}{n+1}\right]_0^1$

$=\dfrac{1}{2}-\dfrac{1}{n+1}=\dfrac{n-1}{2(n+1)}$



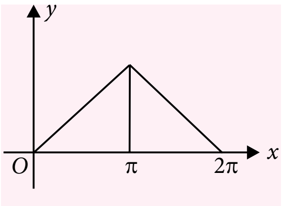
Thus $A_2 \cdot A_3 \cdot A_4 \cdot \ldots\ldots A_n$

$=\dfrac{1}{2^{n-1}}\left(\dfrac{1}{3}\cdot\dfrac{2}{4}\cdot\dfrac{3}{5}\ldots\ldots\dfrac{n-1}{n+1}\right)$

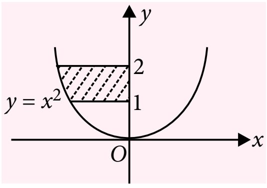
$=\dfrac{1}{2^{n-2}n(n+1)}$

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

2. **(a) :** Obviously area = π^2



3. **(c) :** The required area



$$= -\int_1^2 (-\sqrt{y})\,dy = \left[\frac{2}{3}y^{3/2}\right]_1^2$$

$$= \frac{2}{3}(2\sqrt{2}-1) \text{ sq. units}$$

4. **(b) :** $\frac{1}{2}\int \frac{(\cos^4 x+1)\sin 2x}{\cos 2x}dx$

$$= \frac{1}{8}\int \frac{\left((1+\cos 2x)^2+4\right)\sin 2x}{\cos 2x}dx$$

$$= -\frac{1}{8}\cdot\frac{1}{2}\int \frac{(1+t)^2+4}{t}\,dt$$

$$= -\frac{1}{16}\int\left(t+2+\frac{5}{t}\right)dt$$

$$= -\frac{1}{16}\cdot\left[\frac{1}{2}\cos^2 2x + 2\cos 2x + 5\ln|\cos 2x|\right]+\lambda$$

5. **(c) :** $\int \frac{x^3}{x^4(1+x^4)}dx = \frac{1}{4}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$

$$= \frac{1}{4}\ln\left|\frac{t}{t+1}\right|+c = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right)+c$$

$$= \ln|x| - \frac{1}{4}\ln(1+x^4)+c$$

6. Let $f(x) = ax^3 + bx^2 + cx + d$. Since $f(0)=1$, so $d=1$. Moreover $f'(0) = 0$ *i.e.*, $c = 0$. Also $f''(x) = 6ax + 2b$. Since $f(x)$ doesn't have extremum at 0,

so $f''(0) = 2b = 0$

Hence $f(x) = ax^3 + 1$

$f(1) = 2$

$\Rightarrow a = 1 \Rightarrow f(x) = x^3 + 1$

Now, $\int \frac{f(x)}{\sqrt{x^2+7}}dx = \int \frac{x^3+1}{\sqrt{x^2+7}}dx$

$$= \int \frac{x^3}{\sqrt{x^2+7}}dx + \int \frac{dx}{\sqrt{x^2+7}}$$

$$= \int \frac{(y^2-7)\,y\,dy}{y} + \log|x+\sqrt{x^2+7}|+C$$

(Put $y^2 = x^2 + 7$)

$$= \frac{y^3}{3} - 7y + \log|x+\sqrt{x^2+7}|+C$$

$$= \frac{1}{3}(x^2+7)^{3/2} - 7(x^2+7)^{1/2} + \log|x+\sqrt{x^2+7}|+C$$

7. Let $\sin^{-1}x = \theta$, then $x = \sin\theta$ and $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta|\cos\theta|)$

$= \sin^{-1}(2\sin\theta\cos\theta)$, as $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$= \sin^{-1}(\sin 2\theta)$

$= -\sin^{-1}(\sin(2\theta - \pi))$, $-\frac{\pi}{2} \le 2\theta - \pi \le 0$

$= \pi - 2\theta = \pi - 2\sin^{-1}x$

Hence $\int \sin^{-1}(2x\sqrt{1-x^2})\,dx$

$$= \int(\pi - 2\sin^{-1}x)\,dx = \pi x - 2\int \sin^{-1}x\,dx$$

$$= \pi x - 2\left[x\sin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx\right]$$

$$= \pi x - 2x\sin^{-1}x + \int\frac{2x}{\sqrt{1-x^2}}dx$$

Let $I = \int \frac{2x}{\sqrt{1-x^2}}dx$

$= -\int \frac{2t\,dt}{t}$, Put $1-x^2 = t^2$

$= -2\sqrt{1-x^2} + c$

Hence $\int \sin^{-1}(2x\sqrt{1-x^2})\,dx$

$$= \pi x - 2x\sin^{-1}x - 2\sqrt{1-x^2} + c$$

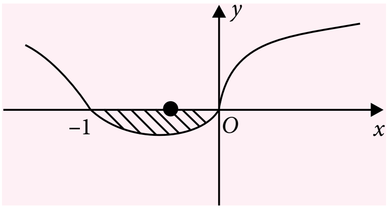
8. Let $f(x) = x^{1/3} + x^{2/3}$

$$\Rightarrow f'(x) = \frac{1+2\cdot x^{1/3}}{3\cdot x^{\frac{2}{3}}} \Rightarrow f''(x) = \frac{-2}{9}\cdot\frac{x^{1/3}+1}{x^{\frac{5}{3}}}$$

Obviously $f(x) = 0$ at $x = 0$, $f(x)>0$ for $x \in (-\infty, -1) \cup (0, \infty)$ and $f(x)<0$ for $x \in (-1, 0)$

Further $f'(x) > 0$, $\forall x \in \left(-\frac{1}{8}, \infty\right)$, where f increases and

$f'(x) < 0$, $\forall x \in \left(-\infty, -\frac{1}{8}\right)$, where f decreases.



$f''(x) > 0,\ \forall\, x \in (-1, 0)$, where f concave up and

$f''(x) < 0,\ \forall\ (-\infty, -1) \cup (0, \infty)$, where f is concave down.

Lastly $f(x) \to \infty\ as\ |x| \to \infty$.

The graph of $y = f(x)$ is shown in adjacent figure.

$$\text{Required area } = -\int_{-1}^{0} \left(x^{\frac{1}{3}} + x^{\frac{2}{3}} \right) dx = \frac{3}{20} \text{ sq.units}$$

9. Let $y_n = d^n y \,/\, dx^n$, then the given differential equation is $y_3 - 8y_2 = 0$ or $y_3/y_2 = 8$ which implies

$\log|y_2| = 8x + C_1$ (integrating both sides)

Putting $x = 0$, we get $C_1 = \log y_2(0) = \log 1 = 0$.

$\therefore$ $\log|y_2| = 8x$ or $y_2 = \pm\, e^{8x}$, *i.e.*, $y_1 = \pm\, e^{8x}/8 + C_2$.

Again putting $x = 0$, we have $C_2 = \pm\, 1/8$.

$$\text{So } y_1 = \pm\frac{1}{8}(e^{8x} - 1) \Rightarrow y = \pm\frac{1}{8}\left(\frac{e^{8x}}{8} - x\right) + C_3$$

Putting $x = 0$, we have

$$C_3 = (1/8) \mp (1/64) = 7/64,\ 9/64.$$

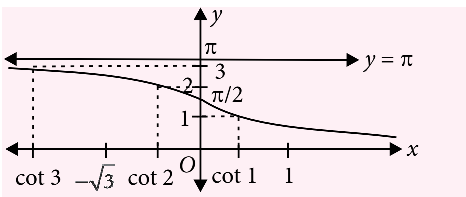
$$\text{Thus } y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$$

$$\text{or } y = -\frac{1}{8}\left(\frac{e^{8x}}{8} - x - \frac{9}{8}\right)$$

10. $f(x) = \cot^{-1} x \Rightarrow f'(x) = \dfrac{-1}{1 + x^2} \Rightarrow f''(x) = \dfrac{2x}{(1 + x^2)^2}$

$\lim\limits_{x \to \infty} f(x) = 0$ and $\lim\limits_{x \to -\infty} f(x) = \pi$. Thus $y = 0$ and $y = \pi$ are asymptotes.

$f(0) = \pi/2 \Rightarrow (0, \pi/2)$ is a point on the graph.



$f'(x) < 0, \forall\, x \Rightarrow f$ is decreasing

$f''(x) < 0,\ \forall\ x < 0$ and $f''(x) > 0,\ \forall\, x > 0$.

Thus $x = 0$ is a point of inflection and f is concave down for $x < 0$, f is concave up for $x > 0$.

$$\text{We have } \frac{\pi}{4} < 1 < \frac{\pi}{2} < 2 < \frac{5\pi}{6} < 3$$

$$\Rightarrow \quad \cot 3 < \cot\frac{5\pi}{6} < \cot 2 < \cot\frac{\pi}{2} < \cot 1 < \cot\frac{\pi}{4}$$

$$\Rightarrow \quad \cot 3 < -\sqrt{3} < \cot 2 < 0 < \cot 1 < 1.$$

$$\Rightarrow \quad [\cot^{-1} x] = \begin{cases} 2, & x \in [-\sqrt{3}, \cot \\ 1, & x \in (\cot 2, \cot 1 \\ 0, & x \in (\cot 1, 1] \end{cases}$$

$$\text{Thus } \int_{-\sqrt{3}}^{1} \left[\cot^{-1} x\right] dx = \int_{-\sqrt{3}}^{\cot 2} 2dx + \int_{\cot 2}^{\cot 1} 1dx + \int_{\cot 1}^{1} 0\, dx$$

$$= 2(\cot 2 + \sqrt{3}) + (\cot 1 - \cot 2) = \cot 2 + \cot 1 + 2\sqrt{3}$$

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. Prove that

$$^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots = \frac{1}{2}\left\{2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right\}$$

– Chinmay, Assam

Ans. $\because$ In given series difference in suffixes is 4.

Now $(1)^{1/4} = (\cos 0 + i\sin 0)^{1/4}$

$= (\cos 2r\pi + i\sin 2r\pi)^{1/4}$

$= \cos\frac{r\pi}{2} + i\sin\frac{r\pi}{2}$, where $r = 0, 1, 2, 3$

Four roots of unity are $1, i, -1$ and $-i$

$= 1, \alpha, \alpha^2$ and α^3 (say)

Also $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

Putting $x = 1, \alpha, \alpha^2$ and α^3, we get

$$2^n = \sum_{r=0}^{n} {}^nC_r \quad \ldots(1), \quad (1+\alpha)^n = \sum_{r=0}^{n} {}^nC_r\alpha^r \quad \ldots(2)$$

$$(1+\alpha^2)^n = \sum_{r=0}^{n} {}^nC_r\alpha^{2r} \quad \ldots(3)$$

and $(1+\alpha^3)^n = \sum_{r=0}^{n} {}^nC_r\alpha^{3r} \ldots(4)$

Multiplying (1) by 1, (2) by α, (3) by α^2 and (4) by α^3 and adding, we get

$2^n + \alpha(1+\alpha)^n + \alpha^2(1+\alpha^2)^n + \alpha^3(1+\alpha^3)^n$

$$= \sum_{r=0}^{n} {}^nC_r\left(1 + \alpha^{r+1} + \alpha^{2r+2} + \alpha^{3r+3}\right) \quad \ldots(5)$$

For $r = 3, 7, 11, \ldots$ R.H.S. of (5) is

$^nC_3(1 + \alpha^4 + \alpha^8 + \alpha^{12}) + {}^nC_7(1 + \alpha^8 + \alpha^{16} + \alpha^{24})$

$+ {}^nC_{11}(1 + \alpha^{12} + \alpha^{24} + \alpha^{36}) + \ldots$

$= 4({}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots) \quad (\because \alpha^4 = 1)$

and L.H.S. of (5)

$= 2^n + i(1+i)^n + i^2(1+i^2)^n + i^3(1+i^3)^n$

$= 2^n + i\{(1+i)^n - (1-i)^n\} = 2^n - 2^{n/2}\cdot 2\sin\frac{n\pi}{4}$

$\therefore 4\left({}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots\right) = 2\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

$\Rightarrow {}^nC_3 + {}^nC_7 + {}^nC_{11} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

2. Prove that $\frac{3}{4} \le \sin^2\theta + \cos^4\theta \le 1$ for all real θ.

– Nandini, Gujarat

Ans. $\sin^2\theta + \cos^4\theta = \cos^4\theta - \cos^2\theta + 1$

$= \left\{(\cos^2\theta)^2 - 2.\frac{1}{2}\cos^2\theta + \frac{1}{4}\right\} + \frac{3}{4}$

$= \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$

Also, $\sin^2\theta + \cos^4\theta = \cos^4\theta - \cos^2\theta + 1$

$= \cos^2\theta(\cos^2\theta - 1) + 1 = 1 - \sin^2\theta\cos^2\theta \le 1$

$\therefore \frac{3}{4} \le \sin^2\theta + \cos^4\theta \le 1.$

3. Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

– Raman, A.P.

Ans. Let

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx = \int \frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$= \int\left(1 - \frac{4}{\pi}\cos^{-1}\sqrt{x}\right)dx = x - \frac{4}{\pi}\int 1\cdot\cos^{-1}\sqrt{x}\,dx$$

$$= x - \frac{4}{\pi}\left[\cos^{-1}\sqrt{x}\cdot x - \int x\cdot\frac{-1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}}dx\right]$$

$$\therefore I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$$

Put $x = \cos^2\theta$; then $dx = -2\cos\theta\cdot\sin\theta\, d\theta$

$$\therefore \int\frac{\sqrt{x}}{\sqrt{1-x}}dx = \int\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}(-2\cos\theta\cdot\sin\theta)d\theta$$

$$= -\int 2\cos^2\theta\, d\theta = -\int(1+\cos 2\theta)d\theta$$

$$= -\left\{\theta + \frac{\sin 2\theta}{2}\right\} + c = -\cos^{-1}\sqrt{x} - \sqrt{x}.\sqrt{1-x} + c$$

$$\therefore I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left\{-\cos^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + c\right\}$$

$$= x + \frac{2}{\pi}(1-2x)\cos^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x(1-x)} + c.$$

Rvp

Rvp